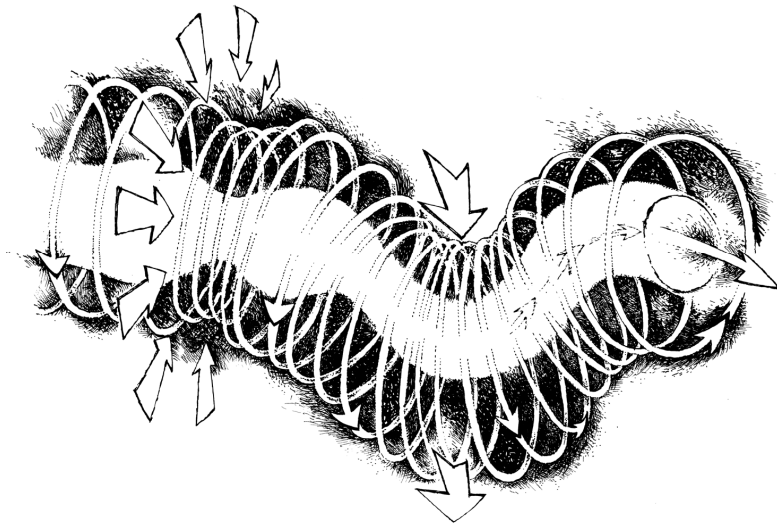# SELECTED TOPICS
IN PLASMA PHYSICS

PETR KULHÁNEK

FOREWORD

In 2011 I published my first book on plasma physics entitled *Introduction to the plasma theory*. It was intended for students of the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague. This book discussed the basic phenomena in plasma physics – from the motions of charged particles to magnetohydrodynamics and the statistical description of plasmas. Although the range of discussed problems was rich, some parts were not covered in the basic course. Therefore I later extended this course to include other topics for doctoral studies – the radiation reaction of a charged particle to its own motion, electromagnetic radiation, turbulence, solitons and other phenomena. The textbook you have just opened is a color online version of these added parts. In contrast to the classical lectures on plasma physics, the individual parts do not build on each other here. After all, the title of the textbook "*Selected topics...*" already suggests this. The English version is partly handled by the translator DeepL. Later, in 2020, a trilogy *Selected Chapters in Theoretical Physics* was published, which includes both the original book and this textbook. There are recordings of all the lectures, for which I owe a great debt of gratitude to Dan Handl, who selflessly recorded my lectures in his spare time.

Additional materials: lecture recordings, current version of the script, applets, charts, etc.:
www.aldebaran.cz

Prague, February 14, 2024
Petr Kulhánek

CONTENTS

1. INTRODUCTORY MATHEMATICS AND PHYSICS

Covariate and contravariant indices

Suppose we have a linear vector space equipped with a basis $\{\mathbf{e}_k\}$. The vector $\mathbf{A}$ can be expanded in this basis into the expression

$$\mathbf{A} = \sum_{k=1}^{N} A^k \mathbf{e}_k = A^k \mathbf{e}_k. \tag{1}$$

The numbers A^k are called components (coordinates, expansion coefficients) of the vector, objects $\mathbf{e}_k$ are elements of the basis. The different positions of the indices indicate that the components of the vectors are transformed differently from the basis elements. From now on we will use the *summation convention*, but the summation will always take place over one subscript (transformed as basis elements) and one superscript (transformed as vector components). Over a pair of the same superscript and subscript, the addition is automatic, these are called dumb indices. The position of the free indices (over which no addition is done) must always remain the same on both sides of the equality. Let's move from one basis to some other, which we mark with wavy lines above the symbols:

$$\{\mathbf{e}_k\} \quad \to \quad \{\tilde{\mathbf{e}}_k\}. \tag{2}$$

Vector $\mathbf{A}$ is an object whose expression cannot depend on the choice of basis, i.e. it must be

$$\mathbf{A} = \tilde{A}^k \tilde{\mathbf{e}}_k = A^k \mathbf{e}_k. \tag{3}$$

The components of the vectors will be transformed between the two bases using matrix $\mathbf{S}$:

$$\tilde{A}^k = S^k{}_l A^l. \tag{4}$$

Note that the summation is over the dummy index l (one is at the top and the other at the bottom). The free index k is on both sides of the equality at the top. Thus, even for matrices, we must distinguish between superscripts and subscripts. Let us denote the transformation matrix of the elements of the basis by $\mathbf{U}$:

$$\tilde{\mathbf{e}}_k = U^l{}_k \mathbf{e}_l. \tag{5}$$

Notice that this is the case in which we sum over one upper and one lower index, the free index k has the same position on both sides of the equality The transformation matrix $\mathbf{U}$ has, like the matrix $\mathbf{S}$, the first index at the top and the second at the bottom. Let us now find out what is the relationship between the two transformation matrices $\mathbf{S}$ and $\mathbf{U}$. Let us start from the expression of the vector $\mathbf{A}$ in the new basis (5):

$$\mathbf{A} = \tilde{A}^k \tilde{\mathbf{e}}_k = S^k{}_l A^l \, U^n{}_k \mathbf{e}_n = U^n{}_k S^k{}_l A^l \mathbf{e}_n.$$

Obviously, in the new basis, the result must be $A^l \mathbf{e}_l$ or $A^k \mathbf{e}_k$, if you like. But there is only one way to achieve this: the last expression must state

$$U^n{}_k S^k{}_l = \delta^n{}_l, \tag{6}$$

where we've marked $\delta^n{}_l$ Kronecker delta. In matrix notation, this condition says that

► $$\mathbf{U} \cdot \mathbf{S} = \mathbf{1}. \tag{7}$$

It is clear that the matrices $\mathbf{U}$ and $\mathbf{S}$ are inverses of each other. This can be seen directly from the decomposition (3) of vector $\mathbf{A}$ into the two bases. If the result is to be the same, the components of the vectors (superscripts) must be transformed "inversely" to the elements of the basis (subscripts). Only in this way will the combinations (3) give a result independent of the choice of basis (vector $\mathbf{A}$). The superscripts will be called contravariant. These indices are transformed in the same way as the components of the vector, i.e. by the transformation

matrix $\mathbf{S}$. We will call the subscripts covariant. These indices are transformed in the same way as the elements of the basis, i.e. using the transformation matrix $\mathbf{U}$. There can be more indices, for example, from the components of two vectors we can construct the expression

$$T^{kl} \equiv A^k B^l \; ; \qquad \tilde{T}^{kl} \equiv S^k{}_o S^l{}_p T^{op} , \qquad (8)$$

which must be transformed as the product of the components of the vectors. Using T^{kl} we can again create a coordinate system independent object, the so-called second order tensor:

$$\ddot{\mathbf{T}} \equiv T^{kl} \mathbf{e}_k \otimes \mathbf{e}_l . \qquad (9)$$

The symbol $\mathbf{e}_k \otimes \mathbf{e}_l$ is called a tensor (dyadic) product, it is an ordered pair of basis elements. We thus understand the expression as an object with components that form the matrix $A^k B^l$:

$$\mathbf{A} \otimes \mathbf{B} = A^k B^l \mathbf{e}_k \otimes \mathbf{e}_l . \qquad (10)$$

Scalar product, raising and lowering of indices

Suppose that a scalar product of two vectors $\mathbf{A} \cdot \mathbf{B}$ is defined on our linear vector space that satisfies the basic properties of a scalar product. If we expand both vectors into a basis, we obtain

$$\mathbf{A} \cdot \mathbf{B} = A^k B^l \; \mathbf{e}_k \cdot \mathbf{e}_l = g_{kl} A^k B^l , \qquad (11)$$

where we have denoted

$$g_{kl} \equiv \mathbf{e}_k \cdot \mathbf{e}_l \qquad (12)$$

the so-called metric coefficients (metrics). We see that the result of the scalar product of two arbitrary vectors can be determined if we know the metric coefficients, i.e. the result of the scalar products of all the elements of the basis among themselves.

Denote the inverse matrix to the metric

$$g^{kl} \equiv \left(g_{kl} \right)^{-1} ; \qquad g^{kl} g_{lm} = \delta^k{}_m . \qquad (13)$$

Let us now introduce auxiliary (dual) objects

$$\mathbf{e}^k \equiv g^{kl} \mathbf{e}_l ; \qquad A_k \equiv g_{kl} A^l . \qquad (14)$$

These are not real basis elements or real vector components, but formal linear combinations given by the metric. It is always the case that the superscript means transforming using the same matrix as the vector components, and the subscript means transforming using the same matrix as the basis elements. Thus, using the metric, we can lower or raise the indices at will, just follow the rule that we add over one superscript and one subscript (this ensures invariance of the sum with respect to the basis transformation). The free indices always keep their position. Let us give an example:

$$g_{lo} T^{klm} = T^k{}_o{}^m .$$

We lowered the middle index using the metric. The scalar product can now be written in several ways:

$$\mathbf{A} \cdot \mathbf{B} = g_{kl} A^k B^l = A^k B_k ,$$

where we lowered the second index using the metric. But we could also have reduced the first index:

$$\mathbf{A} \cdot \mathbf{B} = g_{kl} A^k B^l = A_l B^l = A_k B^k .$$

It is therefore valid

▶
$$\mathbf{A} \cdot \mathbf{B} = g_{kl} A^k B^l = A^k B_k = A_k B^k . \tag{15}$$

The contravariant (upper) component is the real component of the vector, the covariant (lower, dual) component contains the metric. The definition of the inverse metric (13) can also be seen as lowering or raising the indices:

$$
\begin{aligned}
g^{kl} g_{lm} &= \delta^k{}_m ; \\
g^{kl} g_{lm} &= g^k{}_m ;
\end{aligned}
\quad \Rightarrow \quad g^k{}_m = \delta^k{}_m . \tag{16}
$$

Metrics and Kronecker delta are thus the same objects. If both indices are down, they are metric coefficients. If both indices are on top, it is the inverse matrix to the metric coefficients and if the indices are mixed, it is the Kronecker delta, i.e. the elements of the unit matrix. The metric is thus nothing but a unit matrix with appropriately shifted indices. Using tensor notation, we can write

$$\mathbf{1} = \delta^k{}_l \mathbf{e}_k \otimes \mathbf{e}^l = g_{kl} \mathbf{e}^k \otimes \mathbf{e}^l = g^{kl} \mathbf{e}_k \otimes \mathbf{e}_l . \tag{17}$$

Four-vectors, Minkowski metric

In special relativity, we call each quartet of quantities that are transformed by the Lorentz transformation a four-vector. The basic quaternions include the *event* (time and space coordinates of the event), the *four-momentum* (energy and momentum), the *wave four-vector* (angular frequency and wave vector), the *electromagnetic field four-potential* (scalar and vector potentials), the *four-flux* (source terms of Maxwell's equations – charge density and flux) or the *four-gradient*. In the SI system, we must ensure that all 4 components have the same dimension. This can be done most easily by multiplying or dividing the time component by the universal constant c (the speed of light in a vacuum):

▶
$$
x^\mu \equiv \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} ; \qquad
p^\mu \equiv \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix} ; \qquad
k^\mu \equiv \begin{pmatrix} \omega/c \\ \mathbf{k} \end{pmatrix} ;
$$
$$
A^\mu \equiv \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix} ; \qquad
J^\mu \equiv \begin{pmatrix} \rho_Q c \\ \mathbf{j}_Q \end{pmatrix} ; \qquad
\partial_\mu \equiv \begin{pmatrix} \partial/\partial ct \\ \partial/\partial \mathbf{x} \end{pmatrix} .
\tag{18}
$$
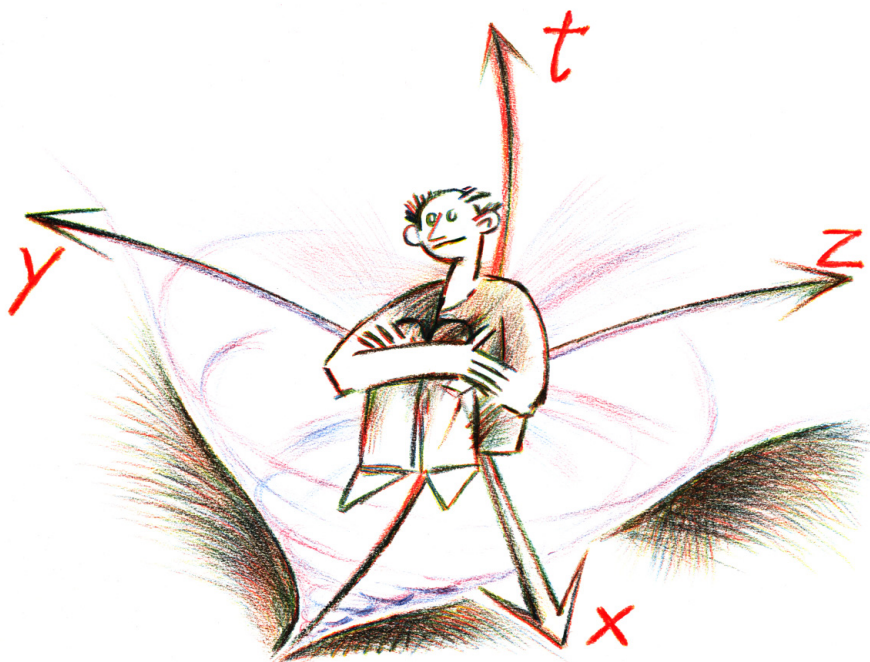
Comments:

1. We will use Greek indices to denote only four-vectors (index 0 corresponds to the time part, indices 1, 2, 3 to the space part).

2. The four-gradient is a covariate (subscript) because

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu},$$

3. so the real components of the vectors are in the denominator, if we write the index in the numerator, it must be in the opposite position, because the transformation matrix changes to the inverse one!

4. The metric in special relativity is called the Minkowski metric. It is diagonal and has a minus in the time part. The same is true for the inverse matrix (metric with superscripts). The mixed-index metric is a unit matrix, i. e. its elements are Kronecker delta:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}; \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix};$$

(19)

$$g^\mu{}_\nu = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}; \quad g_\mu{}^\nu = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}.$$

Simplified, the Minkowski metric is often written as $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, sometimes denoted by $\eta_{\mu\nu}$. Using the metric, we can now easily determine the covariant (dual) components of ordinary four-vectors and the contravariant component of the four-gradient:

▶

$$x_\mu \equiv \begin{pmatrix} -ct \\ \mathbf{x} \end{pmatrix}; \quad p_\mu \equiv \begin{pmatrix} -E/c \\ \mathbf{p} \end{pmatrix}; \quad k_\mu \equiv \begin{pmatrix} -\omega/c \\ \mathbf{k} \end{pmatrix};$$

$$A_\mu \equiv \begin{pmatrix} -\phi/c \\ \mathbf{A} \end{pmatrix}; \quad J_\mu \equiv \begin{pmatrix} -\rho_Q c \\ \mathbf{j}_Q \end{pmatrix}; \quad \partial^\mu \equiv \begin{pmatrix} -\partial/\partial ct \\ \partial/\partial \mathbf{x} \end{pmatrix}.$$

(20)

Examples

Let's find some typical scalar products:

$$k \cdot x = k^\mu x_\mu = k^0(-x_0) + k^1 x_1 + k^2 x_2 + k^3 x_3 = -\omega t + \mathbf{k} \cdot \mathbf{x},$$

on the left is the product of the four-vectors, the last term on the right is the normal product in $\mathbf{R}^3$. Similarly, we determine the results of the other examples

$$\mathrm{d}s^2 \equiv \mathrm{d}x_\mu \mathrm{d}x^\mu = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 ;$$

$$J \cdot A \equiv J_\mu A^\mu = -\rho_Q \phi + \mathbf{j}_Q \cdot \mathbf{A} ;$$

$$\frac{\partial \rho_Q}{\partial t} + \mathrm{div}\, \mathbf{j}_Q = 0 \quad \Leftrightarrow \quad \partial_\mu J^\mu = 0 ;$$

$$\Box f = 0 \quad \Leftrightarrow \quad \partial_\mu \partial^\mu f = 0 .$$

A shorthand notation is often used in which the derivation is written after the comma. The indices before the comma are the actual indices, the indices after the comma are the derivatives:

$$\frac{\partial A^{\mu}}{\partial x^{\nu}} \equiv \partial_{\nu} A^{\mu} \equiv A^{\mu}{}_{,\nu} .$$

This is actually the most economical notation of derivation ever, which makes it obvious at a glance how the derivation is transformed. Let's give some more examples:

$$\frac{\partial \varphi}{\partial x_{\mu}} \equiv \partial^{\mu} \varphi \equiv \varphi^{,\mu} ;$$

$$\frac{\partial \varphi}{\partial x^{\mu}} \equiv \partial_{\mu} \varphi \equiv \varphi_{,\mu} ;$$

$$\Box f \equiv \partial_{\mu} \partial^{\mu} f \equiv f_{,\mu}{}^{\mu} .$$

Lagrange equations for field problems

In classical mechanics we were looking for the dependence of the generalized coordinates $q_k(t)$ on time. For field problems we will look for the space-time dependence of the fields $\varphi_k(t, \mathbf{x})$. Instead of the Lagrange function, we will use the density of the Lagrange function, which depends on time, space, fields and their derivatives:

$$\mathscr{L} = \mathscr{L}(t, x, y, z, \varphi_1, \ldots \varphi_N, \partial \varphi_1/\partial t, \partial \varphi_1/\partial x, \ldots \partial \varphi_N/\partial z) ,$$

which we will abbreviate in the form

$$\mathscr{L} = \mathscr{L}(x^{\mu}, \varphi_k, \varphi_{k,\alpha}) . \tag{21}$$

For the integral of the action, the following will apply

$$S = \int_{\Omega} \mathscr{L}(x^{\mu}, \varphi_k, \varphi_{k,\alpha}) d^3\mathbf{x}\, dt = \int_{\Omega} \mathscr{L}(x^{\mu}, \varphi_k, \varphi_{k,\alpha}) d^4 x . \tag{22}$$

As in mechanics of the bodies, we will look for necessary conditions for the extremality of the action integral, and the variations of the fields will be defined in the same time (but this time also in the spatial coordinate), which will ensure that the variations and partial derivatives are interchangeable. On the boundary of the field $\partial\Omega$ we require the variations to be zero, i. e. the relations as in classical mass point mechanics apply:

$$\delta\varphi_k = \varphi_{k,\mathrm{virt}}(x^{\mu}) - \varphi_{k,\mathrm{real}}(x^{\mu}) ;$$
$$\delta\varphi_k(\partial\Omega) = 0 ; \tag{23}$$
$$\delta\partial_{\mu}\varphi_k = \partial_{\mu}\delta\varphi_k .$$

So let us require that the variation of the action integral is zero:

$$\delta \int_{\Omega} \mathscr{L}(x^{\mu}, \varphi_k, \varphi_{k,\alpha}) d^4 x = 0 . \tag{24}$$

Thanks to the interchangeability of variations and derivations, we can pass a variation into an integral and impress all variables with it (except x^{μ}, it is a variation in the same event):

$$\int_{\Omega} \left[\frac{\partial \mathscr{L}}{\partial \varphi_k} \delta\varphi_k + \frac{\partial \mathscr{L}}{\partial \varphi_{k,\alpha}} \delta\varphi_{k,\alpha} \right] d^4 x = 0 .$$

In the last term, we interchange the variation and derivative: $\delta\varphi_{k,\alpha} = \delta\partial_\alpha\varphi = \partial_\alpha\delta\varphi$ and perform per partes integration (using Gauss' theorem). The integral on the boundary is zero with respect to (23), and therefore we have:

$$\int_\Omega \left[\frac{\partial\mathscr{L}}{\partial\varphi_k} - \partial_\alpha\frac{\partial\mathscr{L}}{\partial\varphi_{k,\alpha}} \right] \delta\varphi_k \, \mathrm{d}^4 x = 0 \,.$$

Since the integration must yield zero for any spacetime domain Ω, the integrand must also be zero (more precisely, *almost everywhere*, i. e., except for sets of dimension less than 4):

$$\left[\frac{\partial\mathscr{L}}{\partial\varphi_k} - \partial_\alpha\frac{\partial\mathscr{L}}{\partial\varphi_{k,\alpha}} \right] \delta\varphi_k = 0 \,. \tag{25}$$

If the fields φ_k are independent, the coefficients of the linear combination (25) will be zero (the whole expression has the form $\Sigma c_k \delta\varphi_k = 0$), i. e.

$$\frac{\partial\mathscr{L}}{\partial\varphi_k} - \partial_\alpha\frac{\partial\mathscr{L}}{\partial\varphi_{k,\alpha}} = 0 \,.$$

We adapt the expression to the standard form of the Lagrange equations

▶ $$\partial_\alpha\left[\frac{\partial\mathscr{L}}{\partial\varphi_{k,\alpha}} \right] - \frac{\partial\mathscr{L}}{\partial\varphi_k} = 0 \,; \qquad k = 1,\ldots,N \,. \tag{26}$$

Unlike the Lagrange equations for solid bodies and mass points, there is not only a time derivative in the first term, but there are derivatives according to all four variables. Lagrange's equations expanded for a single field have the form:

$$\frac{\partial}{\partial t}\left[\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial t)} \right] + \frac{\partial}{\partial x}\left[\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial x)} \right] + \frac{\partial}{\partial y}\left[\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial y)} \right] + \frac{\partial}{\partial z}\left[\frac{\partial\mathscr{L}}{\partial(\partial\varphi/\partial z)} \right] - \frac{\partial\mathscr{L}}{\partial\varphi} = 0 \,. \tag{27}$$

Comment: The Lagrange function is not uniquely determined. Function

$$\tilde{\mathscr{L}} \equiv \mathscr{L} + \partial_\mu K^\mu \tag{28}$$

leads to the same field equations for any four-vector K^μ. This arbitrariness can be used to construct the most "elegant" possible Lagrangian.

◖ **Example:** Let us find the Lagrange equations for the simplest Lagrange function of a scalar field φ containing only the derivatives of this field:

$$\mathscr{L} = \left(\partial_\mu\varphi\right)\left(\partial^\mu\varphi\right). \tag{29}$$

Solution:
The Lagrange function is a scalar (this provides one superscript and one subscript, the expression does not change when the basis/coordinate system changes). If we decompose the Lagrange function, we have:

$$\mathscr{L} = -\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2. \tag{30}$$

After performing all derivations, the Lagrange equation (27) gives

$$-\frac{2}{c^2}\frac{\partial^2\varphi}{\partial t^2} + 2\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial^2\varphi}{\partial y^2} + 2\frac{\partial^2\varphi}{\partial z^2} = 0 \qquad \Rightarrow \qquad -\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \,.$$

Thus, the simplest version of the Lagrange scalar field function leads to the wave equation.

▶

In the Lagrange function (29) the coefficient is usually written ½:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) \quad \Rightarrow \quad \Box\varphi = 0. \tag{31}$$

There are two reasons. Lagrange's equations give the wave equation (without the need to shorten by a factor of 2) and the whole expression (31) is, except for the sign of the time part, the analogue of the kinetic energy (the semiquadratic form of the square of the derivatives). The minus in the time part means that the density of the Lagrange function written in this way corresponds in fact to the negative kinetic energy. The second simplest Lagrange function is obtained by adding a potential term. The quadratic term will lead to a linear partial differential equation (similar to the harmonic oscillator):

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) + \frac{1}{2}\kappa^2 \varphi^2 \quad \Rightarrow \quad (\Box - \kappa^2)\varphi = 0. \tag{32}$$

The Lagrange function of this field is quadratic in derivatives and in the field itself. The second term would correspond to the potential energy density in classical mechanics, the first to the negative kinetic energy density. The resulting Lagrange equation is linear and is a suitable equation, for example, for plasma waves or for the quantum description of particles with spin zero. It is the well-known Klein-Gordon equation.

Canonically associated array

Similarly as we have previously introduced the canonically associated momentum and then energy to a given generalized coordinate, we can define the canonically associated field and energy density in the continuous case by the relations

$$\mathcal{P}_k(t, \mathbf{x}) \equiv \frac{\partial \mathcal{L}}{\partial \varphi_{k,t}} ; \tag{33}$$

$$\mathcal{E}(t, \mathbf{x}) \equiv \sum_k \left(\frac{\partial \mathcal{L}}{\partial \varphi_{k,t}} \varphi_{k,t} \right) - \mathcal{L} . \tag{34}$$
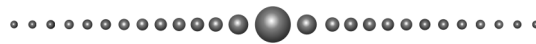
2. CLASSICAL ELECTRODYNAMICS

FOUR-POTENTIAL ANF FIELD TENSOR

The electromagnetic field is usually described by Maxwell's equations

$$\operatorname{div} \mathbf{B} = 0 , \tag{35}$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} , \tag{36}$$

$$\operatorname{div} \mathbf{D} = \rho_{\mathrm{Q}} , \tag{37}$$

$$\operatorname{rot} \mathbf{H} = \mathbf{j}_Q + \frac{\partial \mathbf{D}}{\partial t} , \tag{38}$$

which we complete with material relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} , \tag{39}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) , \tag{40}$$

where vector **P** is the polarization of the environment (electric dipole moment density) and **M** is the magnetization (magnetic dipole moment density). The different style of parentheses of the electric and magnetic quantities is due to historical reasons. Equation (35) implies the existence of a function $\mathbf{A}(t, \mathbf{x})$ such that

▶ $$\mathbf{B} = \operatorname{rot} \mathbf{A} . \tag{41}$$

The equation is then satisfied automatically because the div rot of each function is zero. The quantity **A** is called the vector potential. Substituting expression (41) into equation (36), we obtain the relation

$$\operatorname{rot} \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 ,$$

which implies the existence of a function ϕ such that $\mathbf{E} + \partial \mathbf{A}/\partial t = -\nabla \phi$. Equation (36) is again satisfied automatically and we have an expression for the electric field

▶ $$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} . \tag{42}$$

The function ϕ is called the scalar potential; in the case of stationary fields, (42) becomes the well-known relation

$$\mathbf{E} = -\nabla \phi , \tag{43}$$

the minus sign just reflects the physical fact that the applied force is directed to the minimum potential energy. The electromagnetic field can thus be described by just four quantities – the scalar and vector potential. These four quantities form a relativistic four-vector, see (18),

▶ $$A^\mu = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix} . \tag{44}$$

If we know the four varables A^μ, we can easily determine the electric and magnetic fields from relations (42) and (41). Thus the vectors **E** and **B** are in a sense preferred to the vectors **D** and **H**, since we can determine them directly from the potentials. The preference of **E** and **B** over **D** and **H** also follows from the Lorentz equation of motion, where only the vectors **E** and **B** stand out, and we can even define them from the force action using the Lorentz equation of motion. The electromagnetic field is a derivative of the potentials, both relations (42) and (41) can be simply written using the electromagnetic field tensor

▶
$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & E^x/c & E^y/c & E^z/c \\ -E^x/c & 0 & B^z & -B^y \\ -E^y/c & -B^z & 0 & B^x \\ -E^z/c & B^y & -B^x & 0 \end{pmatrix}. \tag{45}$$

It is a second order antisymmetric tensor which has only six independent components (these are the electric and magnetic fields). The field components can be easily subtracted from the corresponding positions in the tensor.

The potentials are not uniquely determined; two different potentials may correspond to the same electromagnetic field. If we introduce new, transformed potentials using the so-called *gradient transformation*

$$\tilde{A}^\mu \equiv A^\mu + \partial^\mu f, \tag{46}$$

where *f* is a completely arbitrary twice continuously differentiable function, the field does not change:

$$\tilde{F}^{\mu\nu} = \partial^\mu \left(A^\nu + \partial^\nu f \right) - \partial^\nu \left(A^\mu + \partial^\mu f \right) = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}. \tag{47}$$

This arbitrariness in potentials can be used to advantage in formulation of the simplest possible version of Maxwell's equations in potentials.

Maxwell's equations in potentials

Maxwell's equations (35) and (36) were used to introduce the electromagnetic field potentials. Using the electromagnetic field tensor, the remaining two equations with source terms can be easily rewritten in the form

▶
$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\mu, \tag{48}$$

where the four-vector J^μ presents sources of electric and magnetic fields

$$J^\mu \equiv \begin{pmatrix} \rho_Q c \\ \mathbf{j}_Q \end{pmatrix}. \tag{49}$$

This form of Maxwell's equations is obviously relativistic. Let us rewrite Maxwell's equations in the form (48) using potentials:

$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\mu \ ;$$

$$\partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \mu_0 J^\mu \ ;$$

$$\partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu A^\mu = \mu_0 J^\mu. \tag{50}$$

The second term on the left-hand side is the d'Alambert wave operator applied to the field four-potential, while the right-hand side obviously presents the sources of the fields. The only "flaw in the beauty" of the equations written in potentials is the first term. Here we take advantage of the large arbitrariness in the potentials given by the calibration transformation. Let us assume that the quantity $\partial_\nu A^\nu$ is equal to some function of time and space $F(t, \mathbf{x})$:

$$\partial_\nu A^\nu = F(t, \mathbf{x})$$

and, using the gradient transformation (46), choose another four-potential for which we will require

$$\partial_\nu \tilde{A}^\nu = 0 \quad \Rightarrow \quad \partial_\nu A^\nu + \partial_\nu \partial^\nu f = 0 \quad \Rightarrow \quad F(t, \mathbf{x}) + \Box f = 0.$$

Such a gradient transformation will always exist. The function f that generates the transformation need only be chosen to satisfy the equation

$$\Box f = -F(t, \mathbf{x}). \tag{51}$$

In the new potentials, the first term in equation (50) is zero and Maxwell's equations take the simple form

$$\Box \tilde{A}^{\mu} = -\mu_0 J^{\mu}; \tag{52}$$

$$\partial_{\mu} \tilde{A}^{\mu} = 0. \tag{53}$$

These are the wave equations for the four-potential A^{μ}, where the source terms are the components of the four-vector J^{μ}. The equations are supplemented by the Lorenz calibration condition (53). We have shown that the arbitrariness of the potentials can be used to satisfy the Lorenz calibration condition. Even the requirement to satisfy it, does not uniquely determine the potentials! It is clear from equation (51) that the function f is not uniquely determined and any solution of the wave equation can be added to it

$$\Box f_0 = 0. \tag{54}$$

Therefore another gradient transformation is possible

$$\tilde{\tilde{A}}^{\mu} = \tilde{A}^{\mu} + \partial^{\mu} f_0,$$

which can be used, for example, to make zero the scalar potential. We conclude this part by stating that it is always possible to choose potentials such that Maxwell's equations have a simple form

▶
$$\Box A^{\mu} = -\mu_0 J^{\mu};$$
$$\partial_{\mu} A^{\mu} = 0. \tag{55}$$

Fields or potentials? AB experiment

In classical electrodynamics, the electromagnetic field can be described either by means of electromagnetic intensity and magnetic induction or by means of four-potentials. Each of these descriptions has advantages and disadvantages:

1. Electric and magnetic fields are directly measurable by instruments, potentials are not. This situation gives the impression that fields are real quantities, while potentials are just auxiliary mathematical objects.

2. Electric and magnetic fields are unambiguous, but there are infinitely many potentials to a given problem. This can be used to construct the simplest possible equations for the potentials. On the other hand, the ambiguity of potentials again gives the impression that the concept of potentials is just an auxiliary mathematical construction.

3. Maxwell's equations in potentials are simpler, leading to a wave equation with a non-zero right-hand side, for which there are a number of possible solutions. However, after finding the potentials, we still have to determine the field from the formulas $\mathbf{B} = \text{rot } \mathbf{A}$, $\mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t$.

4. Potentials are more appropriate when transforming fields to another coordinate system. They form a four-vector which is transformed by the Lorentz matrix. The transformation of the fields itself is somewhat complicated and is given by the transformation of the field tensor.

5. At first sight, the field four-potential field $(\phi, \mathbf{A})$ fits better into the four-dimensional world of relativity than the six values of $\mathbf{E}$ and $\mathbf{B}$. These are in fact part of the field tensor $F_{\mu\nu}$, which is antisymmetric and has just 6 independent components.

In classical electrodynamics, the two descriptions are completely equivalent and one of them cannot be preferred to the other. A particle can only change its velocity due to electric and magnetic fields. If the fields are zero and the potentials are non-zero (such a situation can occur), no forces act on the particle. In quantum mechanics, the situation is different. The mere presence of a non-zero potential changes the phase of the wave function even if the fields themselves are zero (for example, in the space outside a long coil, the magnetic field is zero and the vector potential is non-zero). The change in the phase of the wave function is reflected by a change in the interference pattern for the two-slit experiment and is therefore a measurable phenomenon. In this sense, classical Maxwell's electrodynamics, supplemented by the Lorentz equation of motion, is an incomplete description of nature, since it does not capture all of the measurable processes occurring in nature. Moreover, field potentials are not just a mathematical construct, but have a real physical impact on the motion of charged particles given by quantum laws. This fact was first pointed out by English theorists Werner Ehrenberg and Raymond Siday in 1949, but their work has not been sufficiently disseminated. A similar phenomenon was again predicted a decade later, in 1959, by Israeli physicist Yakir Aharonov and American-English theorist David Bohm. The Aharonov-Bohm effect (AB effect) was not experimentally confirmed until 1986 by Japanese physicist Akira Tonomura.

Let us first consider a two-slit experiment with the setup shown in the following figure on the left. Behind the slits is a narrow band of non-zero magnetic field (of thickness Δl), which is directed perpendicular to the electron motion. This field will exert a Lorentz force on them

$$F = evB \tag{56}$$

upwards. Since the thickness of the non-zero field layer is small, we will assume that the electron beam moves upwards due to the constant acceleration $a = F/m$ for the time $\Delta t = \Delta l / v$. In the vertical direction, the electrons will be deflected by a distance

$$\Delta y \cong \frac{1}{2} a (\Delta t)^2 = \frac{1}{2} \frac{evB}{m} \left(\frac{\Delta l}{v} \right)^2 = \frac{eB}{2mv} \Delta l^2. \tag{57}$$

The deflection angle of the beam will be according to the classical calculation

$$\operatorname{tg} \alpha \cong \frac{\Delta y}{\Delta l} = \frac{eB \, \Delta l}{2mv}. \tag{58}$$

Whatever the pattern of electron impact on the screen, it should be angularly shifted upwards by the magnetic field by the angle α given by (58).
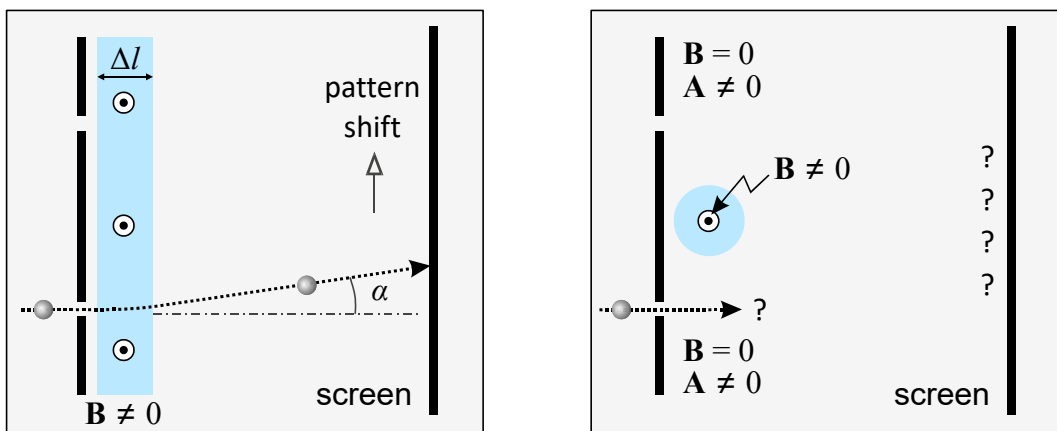
To the explanation of the Aharonov-Bohm phenomenon

Let us now consider a two-slit experiment with the setup as shown on the right. Behind the slots is a long solenoid in which there is a non-zero magnetic field. Outside the solenoid, in the region through which the electrons move, the magnetic field is zero and the pattern should

not be shifted. Only the vector potential is non-zero here. However, the four-potential appears in both the Lagrange and Hamiltonian functions and from here it enters the solution of the Schrödinger time equation. The wave function will take the form (the derivation can be found in quantum theory textbooks)

$$\psi = A \exp\left[\frac{iQ}{\hbar} \int A_\mu \mathrm{d}x^\mu\right],$$
(59)

and the probability of the electron occurrence will be affected by the presence of a non-zero potential and the pattern should be shifted even outside the solenoid, i. e. in a zero magnetic field environment.

In the region through which the electrons pass in the Aharonov-Bohm thought experiment, although the magnetic field is zero, the vector potential is non-zero, which causes a change in the phase of the wave function and thus a shift in the interference pattern.

Japanese physicist Akira Tonomura first attempted to measure this phenomenon in 1982 using an electron holographic microscope, which can record the phase of electrons (they emit a coherent electromagnetic field) in addition to the intensity of the electron beam. The results were inconclusive for the coil used, as the field leaked outside the coil. Therefore, in 1986, Tonomura used a 6 μm diameter toroid-shaped ferromagnet as the field source. The surface was covered with superconducting niobium, which perfectly shielded the magnetic field. The temperature was kept at 5 K. The shift of the interference fringes between the electron beam passing inside the toroid and the electron beam passing outside the magnet was measured. In these regions there is zero magnetic field but different vector potential. The strips have been shifted by the amount predicted by the Aharonov-Bohm effect. The result of the experiment confirms the fact that at the microscopic level potentials play a primary role in the interaction of a charged microworld object with an electromagnetic field.

Lagrange's formulation of Maxwell's equations

The Lagrange density function for the electromagnetic field has the form

$$\mathscr{L} = \mathscr{L}_{\text{field}} + \mathscr{L}_{\text{int}}$$
(60)

The first part describes the electromagnetic field itself without the presence of particles, the second part describes the interaction of the electromagnetic field with charged particles. The interaction part must be some scalar combination of the four-flux J^μ (describing the particles) and the four-potential A^μ (describing the field). The very simplest scalar version leads to the correct field equations:

▶
$$\mathscr{L}_{\text{int}} = J_\mu A^\mu .$$
(61)

The field part of the Lagrange function corresponds to the kinetic term in point particle mechanics, so it should be constructed somehow from the derivatives of the field A^μ, i. e. it should be somehow composed of the electromagnetic field tensor – ideally the square, as in the case of the kinetic energy of a particle or as in the case of the density of the Lagrangian leading to the wave equation. The simplest scalar is the combination $F_{\mu\nu}F^{\mu\nu}$ and the field part of the Lagrange function should be proportional to this expression. We determine the proportionality constant to get the correct field equations (in this case Maxwell's equations):

▶
$$\mathscr{L}_{\text{field}} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$
(62)

The two parts of the Lagrange density function for electromagnetic fields are

▶
$$\mathscr{L}_{\text{elmg}} = \mathscr{L}_{\text{field}} + \mathscr{L}_{\text{int}} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J_\mu A^\mu$$
(63)

In conclusion, let us verify that the Lagrange field equations yield Maxwell's equations:

$$\partial_\alpha \left[\frac{\partial \mathscr{L}}{\partial A^\beta_{,\alpha}} \right] - \frac{\partial \mathscr{L}}{\partial A^\beta} = 0 \; ;$$

$$-\frac{1}{4\mu_0} \partial^\alpha \left[\frac{\partial \left(F_{\mu\nu} F^{\mu\nu} \right)}{\partial A^{\beta,\alpha}} \right] - J_\beta = 0 \; ;$$

$$-\frac{1}{4} \partial^\alpha \left[2 F_{\mu\nu} \frac{\partial F^{\mu\nu}}{\partial A^{\beta,\alpha}} \right] = \mu_0 J_\beta \; ;$$

$$-\frac{1}{2} \partial^\alpha \left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \frac{\partial}{\partial (\partial^\alpha A^\beta)} \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) \right] = \mu_0 J_\beta \; ;$$

$$-\frac{1}{2} \partial^\alpha \left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \left(\delta^\mu_{\ \alpha} \delta^\nu_{\ \beta} - \delta^\nu_{\ \alpha} \delta^\mu_{\ \beta} \right) \right] = \mu_0 J_\beta \; ;$$

$$-\frac{1}{2} \partial^\alpha \left[\partial_\alpha A_\beta - \partial_\beta A_\alpha + \partial_\alpha A_\beta - \partial_\beta A_\alpha \right] = \mu_0 J_\beta \; ;$$

$$-\partial^\alpha F_{\alpha\beta} = \mu_0 J_\beta \quad \Rightarrow \quad -F^{\alpha\beta}_{\ ,\alpha} = \mu_0 J^\beta \quad \Rightarrow \quad F^{\beta\alpha}_{\ ,\alpha} = \mu_0 J^\beta \; ,$$

which are Maxwell's equations of the form (48).

Retarded and advanced potentials

Consider a point charge located at position $\mathbf{r}'$ and an observer at $\mathbf{r}$. The electric field potential will be given by Coulomb's law

$$\phi = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|\mathbf{r} - \mathbf{r}'|}. \tag{64}$$

If the charge is continuously distributed in space with density ρ_Q, the potential will be given by the superposition, i. e. the sum of the potentials from all charge elements:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_Q(\mathbf{r}') \mathrm{d}^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{65}$$

The relation found must also be a solution of the Laplace equation of electrostatics:

$$\Delta\phi = -\frac{\rho_Q}{\varepsilon_0} \quad \Rightarrow \quad \phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_Q(\mathbf{r}') \mathrm{d}^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{66}$$

Those readers who know the technique of the Green's function can find a solution in other way than by using the superposition principle. First, you will solve the Laplace equation with the Dirac impuls on the right-hand side to obtain the Green's function. The solution to the problem will then be the convolution of this Green's function with the true right-hand side of the equation:

$$\Delta G = \delta(\mathbf{r}') \quad \Rightarrow \quad G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|}. \tag{67}$$

$$\Delta\phi = -\frac{\rho_Q}{\varepsilon_0} \quad \Rightarrow \quad \phi(\mathbf{r}) = \left(-\frac{\rho_Q}{\varepsilon_0} \right) * G = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_Q(\mathbf{r}') \mathrm{d}^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \tag{68}$$

The result is of course the same. In the case of electrodynamics, the direct generalization is

$$\Box\phi = -\frac{\rho_Q}{\varepsilon_0} \qquad \Rightarrow \qquad \phi(t,\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_Q(t-\Delta t,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{69}$$

The potential is calculated in retarded time, i. e. the time of the observer minus the propagation time of the signal from the source:

$$\Delta t = |\mathbf{r}-\mathbf{r}'|/c \tag{70}$$

The acquired potential is called the retarded (delayed) potential. Its value is calculated from the distribution of charges in the cone of the observer's past. But this is not the only solution to the wave equation with a right-hand side. In fact, the wave equation itself is immune to the substitution $c \to -c$, therefore, according to (70), it is also satisfied by the solution

$$\phi_+(t,\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(t+\Delta t,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{71}$$

Such a solution is unphysical, it comes as if from the future, i. e. it depends on the distribution of charges in the cone of the observer's future and therefore does not satisfy the principle of causality (the cause must precede the effect in all coordinate systems). The solution is called advanced. Both solutions (physical and non-physical) can be written together in a single formula

$$\phi_\pm(t,\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(t\pm\Delta t,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{72}$$

Similarly, the solution of Maxwell's equations can be written for the whole four-potential

▶ $$\Box A^\alpha = -\mu_0 J^\alpha \qquad \Rightarrow \qquad A^\alpha_\mp(t,\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{J^\alpha(t\mp\Delta t,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{73}$$

It remains to be verified whether the solution found satisfies the Lorenz calibration condition. Let us determine the left-hand side of this condition:

$$\partial_\alpha A^\alpha_\mp = \frac{\mu_0}{4\pi}\int\frac{\partial_\alpha J^\alpha(t\mp\Delta t,\mathbf{r}')\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{74}$$

The calibration condition will be satisfied if

$$\partial_\alpha J^\alpha = 0, \tag{75}$$

which is the continuity equation for electric charge, the law of conservation of charge. It follows immediately from Maxwell's equations

$$\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha \qquad \Rightarrow \qquad \partial_\alpha\partial_\beta F^{\alpha\beta} = \mu_0\partial_\alpha J^\alpha. \tag{76}$$

The left-hand side is the summation of the symmetric and antisymmetric tensor, and is therefore zero, and therefore relation (75) holds. We have computed the potentials for the observer at location $\mathbf{r}$. The charges are localized at the locations $\mathbf{r}'$ over which we integrate. Quite far from the sources, it is possible to perform a classical multipole expansion and determine the radiation fields generated by the moving particles. If we are interested in the reaction of a particle to its own field, we have to do the opposite and detect the fields generated in close proximity to the particle under study.

3. CHARGED PARTICLE EQUATION OF MOTION

Lorentz equation of motion

Let us first introduce the proper time of the particle $d\tau$ as the time passing directly at the particle. For the relativistic interval at the particle location we have

$$ds^2 = dx_\alpha dx^\alpha = -c^2 dt^2 + d\mathbf{x}^2 = -c^2 d\tau^2 . \tag{77}$$

So there is a relationship between proper and ordinary time

$$-c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \quad \Rightarrow \quad d\tau = \sqrt{1 - v^2/c^2}\ dt = \frac{dt}{\gamma} . \tag{78}$$

The proper time is an invariant that we use both in the construction of the Lagrange function of the free particle L_P and in the introduction of four-velocity and four-momentum. For a charged particle, the Lagrange function will consist of a free particle Lagrangian and an interaction Lagrangian:

$$L = L_P + L_{int} . \tag{79}$$

The differential of the action of a free particle should be invariant, i. e. it should be

$$dS = L_P dt \sim d\tau , \tag{80}$$

that is

$$L_P = \alpha \frac{d\tau}{dt} = \alpha \sqrt{1 - v^2/c^2} . \tag{81}$$

The constant α is chosen so that the expression for small velocities becomes the classical relation for kinetic energy, i. e.

▶
$$L_P = -m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} . \tag{82}$$

We determine the interaction term from the density of the Lagrange function (61) as

$$L_{int} = \int \mathcal{L}_{int}(\mathbf{r}') d^3\mathbf{r}' = \int J_\mu(\mathbf{r}') A^\mu(\mathbf{r}') d^3\mathbf{r}' , \tag{83}$$

where the four-flux generated by the point particle localized in the place $\mathbf{r}$ will be

$$J^\mu = \begin{pmatrix} \rho_Q c \\ \mathbf{j}_Q \end{pmatrix} = \begin{pmatrix} Qc\delta(\mathbf{r}' - \mathbf{r}_0) \\ Q\mathbf{v}\delta(\mathbf{r}' - \mathbf{r}_0) \end{pmatrix}; \quad A^\mu = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix} . \tag{84}$$

After simple integration we have

▶
$$L_{int} = -Q\phi + Q\,\mathbf{A} \cdot \mathbf{v} . \tag{85}$$

The first part is, as usual for conservative fields, the negative potential energy of the particle in the electric field. The total Lagrange function is therefore

▶
$$L = L_P + L_{int} = -m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} - Q\phi + Q\mathbf{A} \cdot \mathbf{v} . \tag{86}$$

In standard courses, see for example [1], the corresponding equation of motion for a charged particle is derived straightforwardly from this Lagrange function

▶
$$\frac{d}{dt}(m_0 \gamma \mathbf{v}) = Q\mathbf{E} + Q\,\mathbf{v} \times \mathbf{B} , \tag{87}$$

which is the well-known Lorentz equation of motion with the Lorentz factor on the left side

$$\gamma \equiv \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{88}$$

This is, of course, unpleasant for numerical simulations at high velocities where the Lorentz factor is significant. For simulations it is more convenient to use Hamilton's equations, which are solved with respect to the highest (first) derivative. If we want to use the Lorentz equation of motion, a suitable substitution is

$$\mathbf{u} = \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}, \tag{89}$$

which converts the equation into a form more convenient for numerical integration

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0 \mathbf{u}) = Q\mathbf{E} + \frac{Q\mathbf{u} \times \mathbf{B}}{\sqrt{1 + \mathbf{u}^2/c^2}}. \tag{90}$$

Now we rewrite the Lorentz equation of motion into a relativistic covariant form. To do this, we introduce the four-velocity as

$$\blacktriangleright \qquad\qquad U^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}. \tag{91}$$

The time derivative must be according to the proper time, otherwise the transformation properties of the four-vector would be changed. Using relation (78), the four-velocity can be expressed in terms of the normal velocity $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ as

$$U^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}. \tag{92}$$

For the square of the magnitude of the four-speed, the following holds

$$U^\alpha U_\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = \frac{-c^2\mathrm{d}\tau^2}{\mathrm{d}\tau^2} = -c^2. \tag{93}$$

The scalar product of the four-velocity with itself is indeed an invariant independent of the choice of the coordinate system. Similarly, we can introduce the four-acceleration

$$a^\alpha = \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} \tag{94}$$

The four-acceleration is always perpendicular to the four-velocity, which is easily obtained by deriving the relation (93)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(U^\alpha U_\alpha\right) = 0,$$

$$\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau}U_\alpha + U^\alpha \frac{\mathrm{d}U_\alpha}{\mathrm{d}\tau} = 0,$$

$$2\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau}U_\alpha = 0,$$

From here it immediately follows

$$a^\alpha U_\alpha = 0. \tag{95}$$

The last quantity we will need is the four-momentum defined by the relation

▶
$$p^\alpha \equiv m_0 U^\alpha = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \mathbf{v} \end{pmatrix}. \tag{96}$$

The temporal part of the four-momentum must have the meaning of energy and the spatial part must have the meaning of momentum vector (18), i. e.

$$\begin{pmatrix} \mathcal{E}/c \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \gamma m_0 c \\ \gamma m_0 \mathbf{v} \end{pmatrix} \tag{97}$$

It's easy to get relationships from here

$$\mathcal{E} = \gamma m_0 c^2 = mc^2 \,, \tag{98}$$

$$\mathbf{p} = \gamma m_0 \mathbf{v} = m \mathbf{v} \,, \tag{99}$$

where we introduced the so-called movement mass

$$m = \gamma m_0 \,, \tag{100}$$

which is not a very happy designation, in fact it is just a placeholder for the combination of γm_0. We can easily verify that, using the four-momentum (96) and the field tensor (45), the equation of motion (87) can be written in an elegant form

▶
$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = Q F^{\alpha\beta} U_\beta \,. \tag{101}$$

In the rewriting we used the equality $\mathrm{d}/\mathrm{d}\tau = \gamma\, \mathrm{d}/\mathrm{d}t$. The spatial component gives the Lorentz equation of motion, the time component gives the energy balance

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = Q\,\mathbf{E}\cdot\mathbf{v} \,, \tag{102}$$

which is nothing more than the power delivered by the electric field (force multiplied by velocity). Let us conclude with the key densities of the Lagrange function and the overall Lagrange function for electricity and magnetism

	particle	interaction	field
$\mathscr{L}$	$-\rho_\mathrm{M} c^2 \dfrac{\mathrm{d}\tau}{\mathrm{d}t}$	$J_\alpha A^\alpha$	$-\dfrac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}$
L	$-m_0 c^2 \sqrt{1-\dfrac{\mathbf{v}^2}{c^2}}$	$-Q\phi + Q\mathbf{A}\cdot\mathbf{v}$	$-$

The Lagrange function for the field is meaningless since the field is not localized. If we consider only the particle Lagrange function, we get the equation of motion of a free particle:

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = 0 \,. \tag{103}$$

If we consider the Lagrange functions for the particle and for the interactions, we get the Lorentz equation of motion:

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = Q F^{\alpha\beta} U_\beta \,. \tag{104}$$

If we consider only the Lagrange function for the field, we get Maxwell's equations in vacuum:

$$F^{\alpha\beta}{}_{,\beta} = 0 \,. \tag{105}$$

And if we consider the field and interaction parts of the Lagrange function, we get Maxwell's equations with source terms:

$$F^{\alpha\beta}{}_{,\beta} = \mu_0 J^{\alpha}. \tag{106}$$

Potentials generated by a particle in its vicinity

Consider retarded and advanced potentials generated by a charged particle:

$$A^{\alpha}_{\text{ret}}(t,\mathbf{r}) = A^{\alpha}_{-}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J^{\alpha}(t-\Delta t,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}' \tag{107}$$

$$A^{\alpha}_{\text{adv}}(t,\mathbf{r}) = A^{\alpha}_{+}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J^{\alpha}(t+\Delta t,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}' \tag{108}$$

In summary, we will write both potentials as

$$A^{\alpha}_{\mp}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J^{\alpha}(t\mp\Delta t,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}'. \tag{109}$$

If we are in the close vicinity of the particle, we can assume that the difference between the observation time and the signal transmission time will be small, and therefore we perform Taylor's four-flux expansion in Δt:

$$A^{\alpha}_{\mp}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \int \left(\sum_{k=0}^{\infty} \frac{(\mp\Delta t)^k}{k!} \frac{\partial^k J^{\alpha}(t,\mathbf{r}')}{\partial t^k} \right) \frac{d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{110}$$

For the retarded time we substitute the relation

$$\Delta t = |\mathbf{r}-\mathbf{r}'|/c \tag{111}$$

and we obtain the relation for the four-potential (we assume that the order of summation and integration can be interchanged)

$$A^{\alpha}_{\mp}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{k=0}^{\infty} \left[\frac{(\mp 1)^k}{c^k k!} \frac{\partial^k}{\partial t^k} \int |\mathbf{r}-\mathbf{r}'|^{k-1} J^{\alpha}(t,\mathbf{r}') d^3\mathbf{r}' \right], \tag{112}$$

We divide the expression into even and odd terms:

$$A^{\alpha}_{\mp}(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_{k=0,2,4}^{\infty} \left[\frac{1}{c^k k!} \frac{\partial^k}{\partial t^k} \int |\mathbf{r}-\mathbf{r}'|^{k-1} J^{\alpha}(t,\mathbf{r}') d^3\mathbf{r}' \right] \mp$$
$$\mp \frac{\mu_0}{4\pi} \sum_{k=1,3,5}^{\infty} \left[\frac{1}{c^k k!} \frac{\partial^k}{\partial t^k} \int |\mathbf{r}-\mathbf{r}'|^{k-1} J^{\alpha}(t,\mathbf{r}') d^3\mathbf{r}' \right], \tag{113}$$

The first part is identical for retarded and advanced solutions and has for $k = 0$ a part diverging on the particle's worldline, e. g. Coulombic fields that diverge, but the corresponding force is symmetric with respect to the particle, and therefore the resultant is zero and does not act on the particle. The second part has a different sign for the retarded and the advanced potentials; the solution is different for a wave coming towards the particle and for a wave going away from it. This part does not diverge on the worldline; the resultant is non-zero and corresponds to the reaction of the particle to its own field. The physical solution is a retarded potential, but this can be formally decomposed into symmetric and antisymmetric parts

$$A^{\alpha} = A^{\alpha}_{\text{ret}} = \frac{1}{2}\left(A^{\alpha}_{\text{ret}} + A^{\alpha}_{\text{adv}} \right) + \frac{1}{2}\left(A^{\alpha}_{\text{ret}} - A^{\alpha}_{\text{adv}} \right). \tag{114}$$

For the symmetric part, odd terms disappear and it is given by the first part of the (113):

$$A^\alpha_{sym} \equiv \frac{1}{2}\left(A^\alpha_{ret} + A^\alpha_{adv}\right) = \frac{\mu_0}{4\pi}\sum_{k=0,2,4...}\left[\frac{1}{c^k k!}\frac{\partial^k}{\partial t^k}\int |\mathbf{r}-\mathbf{r}'|^{k-1} J^\alpha(t,\mathbf{r}')\mathrm{d}^3\mathbf{r}'\right]. \quad (115)$$

The antisymmetric part of the potential corresponds to the odd members of the series (113):

$$A^\alpha_{ant} \equiv \frac{1}{2}\left(A^\alpha_{ret} - A^\alpha_{adv}\right) = -\frac{\mu_0}{4\pi}\sum_{k=1,3,5...}\left[\frac{1}{c^k k!}\frac{\partial^k}{\partial t^k}\int |\mathbf{r}-\mathbf{r}'|^{k-1} J^\alpha(t,\mathbf{r}')\mathrm{d}^3\mathbf{r}'\right]. \quad (116)$$

The particle does not diverge on the worldline and the resultant of the corresponding forces acting on the particle is non-zero. This force represents the reaction of the particle to its own fields. It is found in a locally inertial frame that is associated with the particle in some initial phase of motion (Lorentz local coordinate system). In such a system, everything can be solved non-relativistically and, moreover, it is sufficient to take only the first non-zero term (of course, we will have to transform the solution to the general inertial coordinate system eventually, i. e., to covariant form). For the scalar potential, the $k = 1$ term is zero because the integral on the right gives the total charge and its derivative is zero. Therefore, the first non-zero term will be only for $k = 3$:

$$\frac{\phi}{c} = -\frac{\mu_0}{4\pi}\frac{1}{c^3 3!}\frac{\partial^3}{\partial t^3}\int |\mathbf{r}-\mathbf{r}'|^2\,\rho_Q(t,\mathbf{r}')c\,\mathrm{d}^3\mathbf{r}' \quad (117)$$

The first non-zero term for the vector potential will be for $k = 1$:

$$\mathbf{A} = -\frac{\mu_0}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\int \mathbf{j}_Q(t,\mathbf{r}')\mathrm{d}^3\mathbf{r}' \quad (118)$$

For a point particle with world line $x_0{}^\alpha(\tau)$ we have

$$\rho_Q(t,\mathbf{r}) = Q\,\delta(\mathbf{r}'-\mathbf{r}_0),$$
$$\mathbf{j}_Q(t,\mathbf{r}) = Q\,\mathbf{v}\,\delta(\mathbf{r}'-\mathbf{r}_0). \quad (119)$$

The corresponding potentials (near the particle) are after integration are

$$\phi = -\frac{\mu_0 Q}{24\pi c}\frac{\partial^3}{\partial t^3}\left(\mathbf{r}-\mathbf{r}_0(t)\right)^2,$$

► $\quad (120)$

$$\mathbf{A} = -\frac{\mu_0 Q}{4\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t).$$

Abraham-Lorentz equation of motion

The corresponding fields are:
$$\mathbf{B} = \mathrm{rot}\,\mathbf{A} = 0,$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0 Q}{24\pi c}\frac{\partial^3}{\partial t^3}\nabla\left(\mathbf{r}-\mathbf{r}_0(t)\right)^2 + \frac{\partial}{\partial t}\frac{\mu_0 Q}{4\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) =$$

$$= \frac{\mu_0 Q}{24\pi c}\frac{\partial^3}{\partial t^3}2\left(\mathbf{r}-\mathbf{r}_0\right) + \frac{\mu_0 Q}{4\pi c}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \quad (121)$$

$$= -\frac{\mu_0 Q}{12\pi c}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} + \frac{\mu_0 Q}{4\pi c}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \frac{\mu_0 Q}{6\pi c}\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}.$$

The force due to the reaction to the self-field is therefore of the form:

$$\mathbf{F}_{rad} = Q\mathbf{E} = \frac{\mu_0 Q^2}{6\pi c}\dot{\mathbf{a}} \tag{122}$$

This force is called the Abraham-Lorentz force, named after the German physicist Max Abraham (1875–1922), who derived its shape in 1905 [1]. The corresponding equation of motion is called the Abraham-Lorentz (AL) equation of motion:

▶
$$m_0 \mathbf{a} = \mathbf{F}_{ext} + \frac{\mu_0 Q^2}{6\pi c}\dot{\mathbf{a}} \tag{123}$$

In the local Lorentz system of a particle, there is no need to distinguish between rest mass and motion mass, and the equation of motion can be rewritten in the form

▶
$$m_0 \ddot{\mathbf{r}} = \mathbf{F}_{ext} + m_0 \tau_0 \dddot{\mathbf{r}} ; \qquad \tau_0 \equiv \frac{\mu_0 Q^2}{6\pi c m_0}. \tag{124}$$

The constant τ_0 has a time dimension; we will discuss its interpretation shortly. For an electron, its value is approximately 10^{-24} s. The three basic problems of the equation are obvious:

I. The third derivative of the position requires a vague additional initial condition

Classical physics defines a state by position and velocity (momentum). The initial position and velocity (momentum) allow the calculation of the integration constants that appear when solving the equation of motion. However, the presence of a third derivative in the equation of motion implies additional integration constants whose meaning is at least obscure and appears redundant from the point of view of classical physics.

II. At zero external force, there are non-physical exponential solutions

One such solution is easy to find – choose a zero external force and integrate equation (124) three times in succession, choosing the first and second integration constants to be zero:

$$\begin{aligned}
\ddot{\mathbf{r}} &= \tau_0 \dddot{\mathbf{r}}, \\
\dot{\mathbf{r}} &= \tau_0 \ddot{\mathbf{r}}, \\
\mathbf{r} &= \tau_0 \dot{\mathbf{r}}, \\
\mathbf{r} &= \mathbf{r}_0\, e^{t/\tau_0}.
\end{aligned} \tag{125}$$

The solution found is a runaway solution. Such a solution is unphysical, because it is not possible that without the action of any forces the particle is accelerated exponentially. Therefore, the Abraham-Lorentz equation also provides solutions that do not correspond to nature. The general solution for the motion of a free particle (without an external force) can be written as

$$\mathbf{r}(t) = \mathbf{c}_0\, e^{t/\tau_0} + \mathbf{c}_1 t + \mathbf{c}_2. \tag{126}$$

It is clear that by choosing $\mathbf{c}_0 = 0$ we can eliminate the unphysical solution in this case and even reduce the number of integration constants to an acceptable number.

III. The Abraham-Lorentz equation provides a causality-breaking solution

For a force that is non-zero from time t_0 (though constant), the solution at time $t < t_0$ depends on the value of the force in the future, which is unacceptable for physics. Such non-causal solutions exist for any external force that is a mere function of time. One of them is the expression

$$\mathbf{a} = \frac{1}{m_0 \tau_0} \int_t^\infty \mathbf{F}_{ext}(t')\, e^{-\frac{(t'-t)}{\tau_0}}\, dt'. \tag{127}$$

We can easily show that (127) is a solution of the AL equation. We differentiate the left and right sides of (127) in terms of time. For the derivation of the right-hand side, we will differentiate each occurrence of the variable t separately, i. e. we will use the theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_C^t f(t,t')\,\mathrm{d}t' = f(t,t) + \int\limits_C^t \frac{\partial f(t,t')}{\partial t}\,\mathrm{d}t'. \tag{128}$$

The result is

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = -\frac{1}{m_0\tau_0}\mathbf{F}_\text{ext}(t) + \frac{1}{m_0\tau_0^2}\int\limits_t^\infty \mathbf{F}_\text{ext}(t')\,\mathrm{e}^{-\frac{(t'-t)}{\tau_0}}\,\mathrm{d}t' \tag{129}$$

The minus sign in the first term was caused by substituting integration limits so that we differentiate by the upper variable limit. We express the integral on the right from (127):

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = -\frac{1}{m_0\tau_0}\mathbf{F}_\text{ext}(t) + \frac{1}{\tau_0}\mathbf{a} \qquad \Rightarrow \tag{130}$$

$$m_0\,\mathbf{a} = \mathbf{F}_\text{ext}(t) + m_0\tau_0\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}, \tag{131}$$

which is the AL equation. So the acceleration (127) is really a solution of the AL equation. However, the value of the acceleration is calculated from the value of the force $\mathbf{F}_\text{ext}(t)$ in the future, and the solution is clearly non-causal. The future force is integrated with the exponentially decreasing weight. Only the values of the force in the near future are important for the acceleration. The characteristic time constant is just time τ_0, hence it is called *preacceleration time*. For an electron this is a value of approximately 10^{-24} .

<p align="center">* * *</p>

The power associated with generating custom fields is

$$P = \mathbf{F}_\text{rad}\cdot\mathbf{v} = m_0\tau_0\,\dot{\mathbf{a}}\cdot\mathbf{v} = m_0\tau_0\left[\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{a}\cdot\mathbf{v}) - \mathbf{a}^2\right]. \tag{132}$$

For a particle with a constant velocity, the power is zero, but when the particle moves periodically, it radiates. The mean value of the first part will be zero in this case, therefore

$$\langle P\rangle = -\frac{\mu_0 Q^2}{6\pi c}\langle a^2\rangle \tag{133}$$

and the particle loses, as expected, energy by radiation, so at least something is in order in the AL equation. After substituting for the acceleration of the oscillatory motion, we even get the correct formula for dipole radiation.

Lorentz-Dirac equation of motion

If we are not in the Lorentz coordinate system locally associated with a particle, the direct covariant generalization of the equation of motion (124) to an arbitrary system is

$$\blacktriangleright \qquad m_0\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = F_\text{ext}^\alpha + m_0\tau_0\left(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} - \frac{a^2}{c^2}U^\alpha\right); \qquad a^2 \equiv a^\beta a_\beta. \tag{134}$$

The second term in the round bracket ensures the correct size of the four-vectors. Try multiplying the equation with the external Lorentz force by U_α:

<p align="center">24</p>

$$m_0 \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} U_\alpha = QF^{\alpha\beta} U_\beta U_\alpha + m_0 \tau_0 \left(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} U_\alpha - \frac{a^2}{c^2} U^\alpha U_\alpha \right)$$

The term on the left is zero because the four-acceleration is perpendicular to the four-velocity according to (95). The first term on the right is also zero because it is the contraction of a symmetric and an antisymmetric tensor. We modify the third term by the derivative of the product, and in the fourth term we express the magnitude of the four-velocity from (93):

$$0 = 0 + m_0 \tau_0 \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \left(a^\alpha U_\alpha \right) - a^\alpha a_\alpha - \frac{a^2}{c^2} (-c^2) \right)$$

The first term in the round bracket is again zero according to (95) and the remaining terms are exactly subtracted:

$$0 = 0 + m_0 \tau_0 \left(0 - a^2 + a^2 \right)$$

Without the last compensating term, the Lorentz-Dirac equation (134) would not be covariant. The equation was derived by the English physicist Paul A. M. Dirac (1902–1984) using considerations of the conservation of the four-momentum of the field-particle system in 1938 [2]. Our derivation is based on a review article [4]. We can modify the compensating term to the most commonly used form (we use the fact that the four-acceleration is always perpendicular to the four-velocity, i. e., $a_\beta U^\beta = 0$)

$$a^2 U^\alpha = a^\beta a_\beta U^\alpha = a^\beta \frac{\mathrm{d}U_\beta}{\mathrm{d}\tau} U^\alpha = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \left(a^\beta U_\beta \right) - \frac{\mathrm{d}a^\beta}{\mathrm{d}\tau} U_\beta \right) U^\alpha = -\frac{\mathrm{d}a^\beta}{\mathrm{d}\tau} U^\alpha U_\beta,$$

and write the Lorentz-Dirac equation in an alternative form, in which the radiation force is proportional to the square of the acceleration (similar to the AL equation)

▶
$$m_0 \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = F^\alpha_{\mathrm{ext}} + m_0 \tau_0 \left(g^\alpha{}_\beta + \frac{U^\alpha U_\beta}{c^2} \right) \frac{\mathrm{d}a^\beta}{\mathrm{d}\tau}. \tag{135}$$

Pauli radiation reaction formula

In the generalized Lorentz-Dirac equation (134), the radiation reaction acting upon a particle has the form

$$F^\alpha_{\mathrm{rad}} = m_0 \tau_0 \left(\frac{\mathrm{d}a^\alpha}{\mathrm{d}\tau} - \frac{a^2}{c^2} U^\alpha \right). \tag{136}$$

We now move from proper time to coordinate time, using the expression of the interval in the proper system of the particle and in the laboratory system, see (78):

$$\mathrm{d}s^2 = -c^2 \mathrm{d}\tau^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \qquad \Rightarrow \tag{137}$$

$$\mathrm{d}\tau = \mathrm{d}t \sqrt{1 - \mathbf{v}^2/c^2}, \tag{138}$$

and that's why we can write

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\mathrm{d}f}{\mathrm{d}t} \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \frac{\mathrm{d}f}{\mathrm{d}t} = \gamma \dot{f}. \tag{139}$$

The dot will denote, as usual, derivations by coordinate time. Instead of the velocity we will use the dimensionless velocity $\boldsymbol{\beta}$, which has the advantage that its value does not depend on the choice of units

$$\boldsymbol{\beta} \equiv \mathbf{v}/c \tag{140}$$

First we determine the time change of the Lorentz contraction factor γ:

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}\right) = \frac{d}{dt}\left(1 - \mathbf{v}^2/c^2\right)^{-1/2} = \frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2}\left(1 - \mathbf{v}^2/c^2\right)^{-3/2} = \gamma^3 \boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}. \tag{141}$$

Now let's calculate the individual terms needed to complete the radiation reaction (136):

$$x^\alpha = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix}; \tag{142}$$

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau} = \gamma\frac{dx^\alpha}{dt} = \gamma\begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} = \gamma c\begin{pmatrix} 1 \\ \boldsymbol{\beta} \end{pmatrix}; \tag{143}$$

$$a^\alpha \equiv \frac{dU^\alpha}{d\tau} = \gamma\frac{dU^\alpha}{dt} = \gamma\frac{d}{dt}\begin{pmatrix} \gamma c \\ \gamma c\boldsymbol{\beta} \end{pmatrix} = \cdots = c\begin{pmatrix} \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) \\ \gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\boldsymbol{\beta} + \gamma^2\dot{\boldsymbol{\beta}} \end{pmatrix}. \tag{144}$$

Note that the derived expression actually has an acceleration dimension. The velocity of light before the expression introduces the dimension $[c]$ = m/s, the time derivative (dot) introduces another 1/s. The perceptive student can still check that $a^\alpha U_\alpha = 0$. Next, we need a^2:

$$a^2 = a^\alpha a_\alpha = -(a^0)^2 + \mathbf{a}\cdot\mathbf{a} = \cdots = c^2\left[\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^4\dot{\beta}^2\right], \tag{145}$$

The last term that occurs in the radiation reaction is $da^\alpha/d\tau$. The calculation is straightforward, but somewhat tedious, and leads to a fairly simple expression

$$\frac{da^\alpha}{d\tau} = \gamma\frac{da^\alpha}{dt} = \cdots = c\begin{pmatrix} 4\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^5\left[\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}} + \dot{\beta}^2\right] \\ 4\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta} + \gamma^5\left[(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}})\boldsymbol{\beta} + 3(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\dot{\boldsymbol{\beta}} + \dot{\beta}^2\boldsymbol{\beta}\right] + \gamma^3\ddot{\boldsymbol{\beta}} \end{pmatrix} \tag{146}$$

By simply combining the last expressions, we obtain the radiation part of the right-hand side of the Lorenz-Dirac equation:

$$\frac{da^\alpha}{d\tau} - \frac{a^2}{c^2}U^\alpha = c\begin{pmatrix} 3\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^5(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}}) \\ 3\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta} + \gamma^5\left[(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}})\boldsymbol{\beta} + 3(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\dot{\boldsymbol{\beta}}\right] + \gamma^3\ddot{\boldsymbol{\beta}} \end{pmatrix} \tag{147}$$

The equation of motion (134) now takes the form

$$\frac{dp^\alpha}{d\tau} = F^\alpha_{\text{ext}} + m_0 c\tau_0\begin{pmatrix} 3\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^5(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}}) \\ 3\gamma^7(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta} + \gamma^5\left[(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}})\boldsymbol{\beta} + 3(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\dot{\boldsymbol{\beta}}\right] + \gamma^3\ddot{\boldsymbol{\beta}} \end{pmatrix}; \tag{148}$$

where p^α is the four-momentum of the particle. After converting the left-hand side into coordinate time, we have for the radiative part

$$\blacktriangleright \qquad \frac{dE_{\text{rad}}}{dt} = m_0 c^2\tau_0\left[3\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2 + \gamma^4(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}})\right]; \tag{149}$$

$$\blacktriangleright \qquad \mathbf{F}_{\text{rad}} \equiv \frac{d\mathbf{p}_{\text{rad}}}{dt} = m_0 c\tau_0\left[3\gamma^6(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})^2\boldsymbol{\beta} + \gamma^4(\boldsymbol{\beta}\cdot\ddot{\boldsymbol{\beta}})\boldsymbol{\beta} + 3\gamma^4(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}})\dot{\boldsymbol{\beta}} + \gamma^2\ddot{\boldsymbol{\beta}}\right]. \tag{150}$$

The last two terms represent the loss of energy (it becomes negative) and momentum from the particle due to its own radiation. The expression (150) was derived in 1958 by Wolfgang Pauli [3] and is widely quoted by the runaway electron community. For ultrarelativistic particles with $\gamma > 1$, the terms in square brackets are ordered by the decreasing power of the Lorentz factor γ.
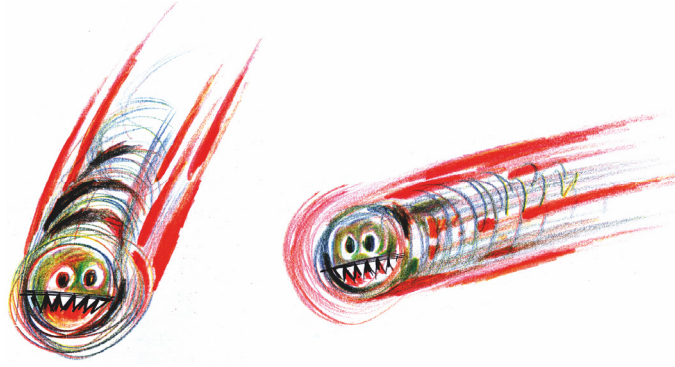
Iterative solution – Landau-Lifshitz equation

An interesting solution to the problem is the assumption that the radiative term in (124) is small, allowing the acceleration to be estimated iteratively:

$$\mathbf{a}_0 = \frac{\mathbf{F}_{\text{ext}}}{m_0} \tag{151}$$

and insert this first iteration into the right-hand side of the equation (124):

$$m\,\mathbf{a}_1 \approx \mathbf{F}_{\text{ext}} + \tau_0 \dot{\mathbf{F}}_{\text{ext}}. \tag{152}$$

This equation has no problems with third derivatives, but sometimes it is not clear under what conditions the first iteration is sufficient and when iterations need to be repeated. Again, the iterative solution only works for a force that explicitly depends on time (e. g., an electron in a laser pulse field). The relativistic generalization of (152) is straightforward:

$$m_0 \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = F_{\text{ext}}^\alpha + \tau_0 \left(g^\alpha{}_\beta + U_\beta U^\alpha \right) F_{\text{ext},\gamma}^\beta U^\gamma. \tag{153}$$

For the Lorentz force, the resulting equation is called the *Landau-Lifshitz equation*:

▶ $$m_0 \frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = QF^{\alpha\beta}U_\beta + \tau_0 \left(g^\alpha{}_\beta + U_\beta U^\alpha \right) QF^{\beta\alpha}{}_{,\gamma} U_\alpha U^\gamma. \tag{154}$$

There are various equivalent rewritings of this equation. The basic condition for its use is that the radiative force must be much smaller than the electromagnetic force. In addition, the time it takes for the signal to pass through the interaction region should be much smaller than the typical time of force field changes. As a rule, these conditions are expressed in the form [7]

$$E \ll \frac{E_S}{\alpha}; \qquad \lambda \gg \alpha\,\lambda_C, \tag{155}$$

where we denote the E_S Schwinger field (the threshold field for the formation of electron-positron pairs from vacuum), λ_C Compton wavelength (the typical change in the wavelength of a photon interacting with an electron), and α fine structure constant (describing the intensity of the electromagnetic interaction). These three key quantities are given by

$$E_S \equiv \frac{m_e^2 c^3}{e\,\hbar} \approx 1.3\times10^{18}\ \text{V/m}, \tag{156}$$

$$\lambda_C \equiv \frac{2\pi\hbar}{m_e c} \approx 2.4\times10^{-12}\ \text{m}, \tag{157}$$

$$\alpha \equiv \frac{e^2}{4\pi\varepsilon_0\,\hbar c} \approx \frac{1}{137}. \tag{158}$$

Efforts to solve problems using integrals or expansions

The solution of equation (124) can be written in a form that completely eliminates the non-physical exponentially increasing solutions. We obtain it from (129) by substituting

$$s = (t' - t)/\tau_0, \tag{159}$$

that leads to a solution

$$\mathbf{a}(t) = \frac{1}{m} \int_0^\infty \mathbf{F}_{\text{ext}}(t + s\tau_0)\, e^{-s}\, ds \tag{160}$$

The problem of causality violation persists. In 2001, American physicists Michael Ibison and Harold Puthoff proposed a solution in the form of a series [5] in parameter τ_0 based on (160):

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{1}{m_0 c \gamma^3} \sum_{n=0}^\infty \left(\gamma \tau_0 \frac{d}{dt} \right)^n \left(\gamma^2 \left[\mathbf{F}_{\text{ext}} - \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{F}_{\text{ext}}) \right] \right);$$

$$\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}; \qquad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}. \tag{161}$$

While this expansion looked promising, it turned out to have two major flaws: 1) it only works for an external time-dependent force (which is not the Lorentz force), 2) for energies higher than the rest energy of the particle the series diverges. More promising is another expansion published by Cuban physicists (Cabo Montes de Oca, Cabo Bizet) in 2015 [6]. However, the time-dependent force problem persists, and moreover, the solution in the form of an infinite series that needs to be further integrated member by member is not ideal.

The general solution of equation (124) for a force depending only on time can be written in another alternative form

$$\mathbf{a}(t) = e^{t/\tau_0} \left[\mathbf{C} - \frac{1}{m_0 \tau_0} \int_{-\infty}^t \mathbf{F}_{\text{ext}}(t')\, e^{-t'/\tau_0}\, dt' \right] \tag{162}$$

For a constant force appearing at time t_0, the $\mathbf{C}$ can be chosen so that the unphysical exponential solution vanishes, but the solution at times $t < t_0$ will depend on the value of the force at $t > t_0$, which is again hardly acceptable. Ambiguities and unanswered questions still remain.

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4. CHARGED PARTICLE RADIATION

Multipole expansion (stationary)

To understand the individual radiative terms, we will need to know the electric and magnetic dipole moments and the electric quadrupole moment. Therefore, we will first perform the multipole development in the stationary case, where everything is simpler, and in the next section we will discuss the radiative expansion. Thus, we will assume that there is no retardation in the equations for the potentials (107) and the charge four-flux is only a function of the spatial coordinates (for example, the particle generated currents do not depend on time)

$$A^\alpha(t,\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J^\alpha(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, d^3\mathbf{r}' \tag{163}$$

Assume that the sources of the fields are localized, forming some clump of particles in the region where we place the origin of the coordinate system:
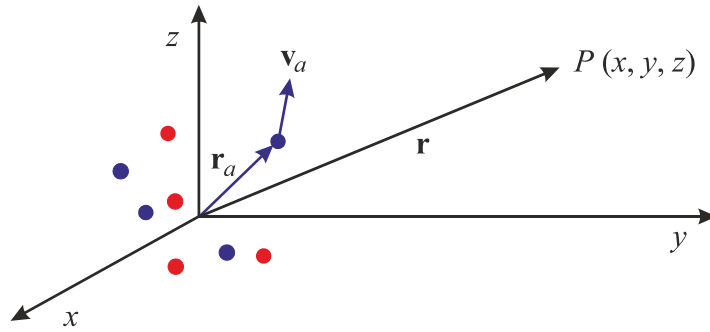
Each of the charges has a position vector $\mathbf{r}_a$, a velocity vector $\mathbf{v}_a$, a mass m_a and a charge Q_a. The observer is at a large distance from the source of the fields and has a position $\mathbf{r}$. The charge four-flux will take the form of a superposition over the positions of the individual particles

$$\rho_Q(\mathbf{r}') = \sum_a Q_\alpha \delta(\mathbf{r}'-\mathbf{r}_a), \tag{164}$$

$$\mathbf{j}_Q(\mathbf{r}') = \sum_a Q_\alpha \mathbf{v}_a \delta(\mathbf{r}'-\mathbf{r}_a). \tag{165}$$

For both potentials in this case we have (after substitution) a simple expression

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_a \frac{Q_a}{|\mathbf{r}-\mathbf{r}_a|}, \tag{166}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sum_a \frac{Q_a \mathbf{v}_a}{|\mathbf{r}-\mathbf{r}_a|}. \tag{167}$$

We observe the field at the far point $\mathbf{r}$, we assume $r \gg r_a$, so we can do Taylor expansion for the argument $\mathbf{r}$ and the increment $\mathbf{r}_a$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_a|} = \frac{1}{r} - x_k^{(a)} \frac{\partial}{\partial x_k}\left(\frac{1}{r}\right) + \frac{1}{2!} x_k^{(a)} x_l^{(a)} \frac{\partial^2}{\partial x_k \partial x_l}\left(\frac{1}{r}\right) \pm \cdots \tag{168}$$

Now we perform the derivations (using $\partial r/\partial x_k = x_k/r$)

$$\frac{1}{|\mathbf{r}-\mathbf{r}_a|} = \frac{1}{r} + x_k^{(a)} \frac{x_k}{r^3} + \frac{1}{2!} x_k^{(a)} x_l^{(a)} \frac{3 x_k x_l - r^2 \delta_{kl}}{r^5} \pm \cdots \tag{169}$$

We insert the derived development into relations (166) and (167) for the potentials and extract the quantities over which no summation takes place:

$$\phi = \frac{1}{4\pi\varepsilon_0 r}\sum_a Q_a \; + \; \frac{x_k}{4\pi\varepsilon_0 r^3}\sum_a Q_a x_k^{(a)} \; + \; \frac{3x_k x_l - r^2\delta_{kl}}{8\pi\varepsilon_0 r^5}\sum_a Q_a x_k^{(a)} x_l^{(a)} \; + \; \cdots \quad (170)$$

$$\mathbf{A} = \frac{\mu_0}{4\pi r}\sum_a Q_a \mathbf{v}_a \; + \; \frac{\mu_0}{4\pi r^3}x_k\sum_a Q_a \mathbf{v}_a x_k^{(a)} \; + \; \cdots \quad (171)$$

In the last written scalar potential development term, the expression in the summation can be modified to a similar form as the expression before the summation:

$$\frac{3x_k x_l - r^2\delta_{kl}}{8\pi\varepsilon_0 r^5}\sum_a Q_a x_k^{(a)} x_l^{(a)} = \frac{3x_k x_l - r^2\delta_{kl}}{24\pi\varepsilon_0 r^5}\sum_a Q_a \left(3x_k^{(a)} x_l^{(a)}\right) =$$

$$= \frac{3x_k x_l - r^2\delta_{kl}}{24\pi\varepsilon_0 r^5}\sum_a Q_a \left(3x_k^{(a)} x_l^{(a)} - r_a^2\delta_{kl}\right). \quad (172)$$

Subtracting the term $r_a^2\delta_{kl}$ gives the tensor in the bracket a zero trace, but nothing actually happens because

$$\left(3x_k x_l - r^2\delta_{kl}\right)r_a^2\delta_{kl} = \left(3r^2 - 3r^2\right)r_a^2 = 0. \quad (173)$$

For the vector potential, the first term after the equation is zero (the sum of the contributions of all electric currents must be zero in the stationary case for an isolated system of charges). In the next term, the resulting combination can be decomposed into symmetric and antisymmetric parts

$$\sum_a Q_a \mathbf{v}_a (\mathbf{r}\cdot\mathbf{r}_a) = \sum_a Q_a \left(\frac{1}{2}\left[\mathbf{v}_a(\mathbf{r}\cdot\mathbf{r}_a) + \mathbf{r}_a(\mathbf{r}\cdot\mathbf{v}_a)\right] + \frac{1}{2}\left[\mathbf{v}_a(\mathbf{r}\cdot\mathbf{r}_a) - \mathbf{r}_a(\mathbf{r}\cdot\mathbf{v}_a)\right]\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\sum_a Q_a \mathbf{r}_a(\mathbf{r}\cdot\mathbf{r}_a) + \frac{1}{2}\sum_a Q_a \left(\mathbf{r}_a\times\mathbf{v}_a\right)\times\mathbf{r}. \quad (174)$$

Due to the assumption of stationarity, the symmetric part is zero (this is no longer true for radiation) and the antisymmetric part gives the double vector product

$$\mathbf{A} = 0 + \frac{\mu_0}{4\pi r^3}\left(\sum_a \frac{1}{2}Q_a(\mathbf{r}_a\times\mathbf{v}_a)\right)\times\mathbf{r} \; + \; \cdots. \quad (175)$$

Overall, then, the first development terms we will need for radiation theory are of the form:

►
$$\begin{aligned}
\phi^{(0)} &= \frac{Q}{4\pi\varepsilon_0 r}\,; & Q &\equiv \sum_a Q_a\,, \\[2mm]
\phi^{(1)} &= \frac{\mathbf{p}_{\mathrm{E}}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}\,; & \mathbf{p}_{\mathrm{E}} &\equiv \sum_a Q_a\mathbf{r}_a\,, \\[2mm]
\phi^{(2)} &= \frac{3x_k x_l - r^2\delta_{kl}}{24\pi\varepsilon_0 r^5}Q_{kl}\,; & Q_{kl} &\equiv \sum_a Q_a\left(3x_k^{(a)} x_l^{(a)} - r_a^2\delta_{kl}\right)\,, \\[2mm]
\mathbf{A}^{(0)} &= 0\,; & Q_{\mathrm{M}} &= 0\,, \\[2mm]
\mathbf{A}^{(1)} &= \frac{\mu_0}{4\pi r^3}\mathbf{p}_{\mathrm{M}}\times\mathbf{r}\,; & \mathbf{p}_{\mathrm{M}} &\equiv \sum_a \frac{1}{2}Q_a(\mathbf{r}_a\times\mathbf{v}_a)\,.
\end{aligned} \qquad (176)$$

The individual contributions are called monopole (0), dipole (1) and quadrupole (2). An electrically neutral system has a total charge Q zero, and thus the entire monopole term is zero. The reader is referred to reference [1] for further properties of the individual multipole development terms, their relation to the polarization and magnetization vectors, and the meaning of dipole moments for simple systems. We note that the electric quadrupole contribution can be written invariantly as:

▶
$$\phi^{(2)} = \frac{3\mathbf{r}\otimes\mathbf{r} - r^2\overset{\leftrightarrow}{\mathbf{1}}}{24\pi\varepsilon_0 r^5} : \overset{\leftrightarrow}{\mathbf{Q}} ; \qquad \overset{\leftrightarrow}{\mathbf{Q}} \equiv \sum_a Q_a\left(3\mathbf{r}_a\otimes\mathbf{r}_a - r_a^2\mathbf{1}\right). \tag{177}$$

## Multipolar expansion (radiative)

Consider now time-varying sources of potentials

$$A^\alpha = \frac{\mu_0}{4\pi}\int \frac{J^\alpha(t',\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' ; \tag{178}$$

$$t' \equiv t - \Delta t = t - \frac{|\mathbf{r}-\mathbf{r}'(t')|}{c}. \tag{179}$$

The situation is now a little more complicated. For one thing, the retarded time is not specified explicitly, but is given by the implicit formula (179); for another, one will have to do Taylor expansion in both the denominator and the numerator, and then select the raditive terms. These decay with distance as $1/r$, where r is the distance of the observer from the origin of the coordinate system, which is localized in the source region. We only consider a field that is able to carry energy to any distance from the source, so the integration of the Poynting vector $\mathbf{E}\times\mathbf{H}$, which is the energy flux, over the solid angle $r^2\mathrm{d}\Omega$ must give a finite contribution at any distance. This fact, in turn, simplifies the problem, since it is sufficient to select only terms decaying as $1/r$ from the expansions of the potentials and fields.
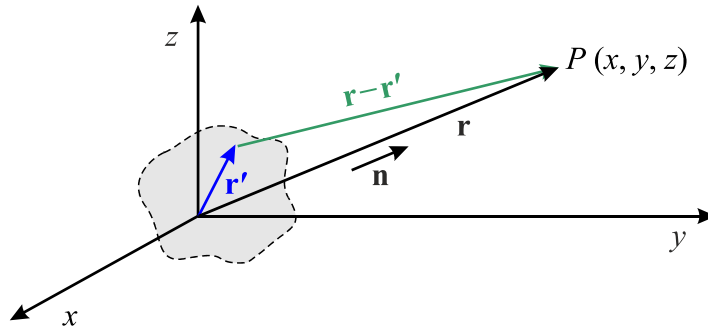
The expansion of the denominator was also carried out in the stationary case, see (168), for the purposes of radiation theory it is sufficient to take only the first term, the others decay faster than $1/r$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r}. \tag{180}$$

In the four-flux argument, it will be necessary to expand a retarded time for an observer located at a large distance from the source

$$t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} = t - \frac{r}{c} - \frac{\partial}{\partial x_k}\left(\frac{r}{c}\right)(-x_k') + \cdots = t - \frac{r}{c} + \frac{1}{c}\frac{x_k}{r}x_k' + \cdots = t - \frac{r}{c} + \frac{\mathbf{n}\cdot\mathbf{r}'}{c} + \cdots$$

The result obtained is quite understandable. The first term is the observer's time, the second term is the retarded time relative to the origin (common to all sources), and the last term represents the subtle nuances of the retardation for the individual elements of the field source. Let us write it down clearly:

$$t' \approx \tau + \frac{\mathbf{n} \cdot \mathbf{r}'}{c} ; \tag{181}$$

$$\tau \equiv t - r/c . \tag{182}$$

We denote by τ the common retarded time with respect to the origin, by $\mathbf{n}$ the normal vector to the distant integration surface (unit vector of the connection of the origin of the system with the observer), $\mathbf{n} = \mathbf{r}/r$. Of course, we did not get rid of the implicit dependence, the vector $\mathbf{r}'$ is again expressed in retarded time. Considering both developments (180) and (181), the relation (178) for the four-potential takes the form

$$A^\alpha = \frac{\mu_0}{4\pi r} \int J^\alpha(\tau + \mathbf{n} \cdot \mathbf{r}'/c, \mathbf{r}') \, d^3\mathbf{r}' . \tag{183}$$

We now develop the four-flow in time τ and increment $\mathbf{n} \cdot \mathbf{r}'$.

$$A^\alpha(\tau) = \frac{\mu_0}{4\pi r} \int J^\alpha(\tau, \mathbf{r}') \, d^3\mathbf{r}' + \frac{\mu_0}{4\pi r} \int \frac{\partial J^\alpha}{\partial \tau} \left(\frac{\mathbf{n} \cdot \mathbf{r}'}{c} \right) d^3\mathbf{r}' + \cdots. \tag{184}$$

We will calculate the electromagnetic fields from the relations

$$\mathbf{E}_{\text{rad}} = -\nabla \phi_{\text{rad}} - \frac{\partial \mathbf{A}_{\text{rad}}}{\partial t} = -\frac{\partial \mathbf{A}_{\text{rad}}}{\partial t} , \tag{185}$$

$$\mathbf{B}_{\text{rad}} = \text{rot} \, \mathbf{A}_{\text{rad}} . \tag{186}$$

Since the spatial derivative of the scalar potential will decay with distance at least as $1/r^2$ (ϕ itself decays as $1/r$), the scalar potential will not contribute to the radiative fields. Thus, in relation (184), we will continue to consider only the spatial part and additionally we extract the derivation by the common retarded time τ before integration:

$$\mathbf{A}(\tau) = \frac{\mu_0}{4\pi r} \int \mathbf{j}_Q(\tau, \mathbf{r}') \, d^3\mathbf{r}' + \frac{\mu_0}{4\pi r} \frac{\partial}{\partial \tau} \int \mathbf{j}_Q(\tau, \mathbf{r}') \left(\frac{\mathbf{n} \cdot \mathbf{r}'}{c} \right) d^3\mathbf{r}' + \cdots. \tag{187}$$

In what follows we will see that the first term describes electric dipole radiation and the second term describes magnetic dipole and electric quadrupole radiation. Let us now perform the integration for a system of charged particles localized in the neighborhood of the origin, for which the current density is

$$\mathbf{j}_Q(\tau, \mathbf{r}') = \sum_a Q_\alpha \mathbf{v}_a(\tau) \delta(\mathbf{r}' - \mathbf{r}_a) . \tag{188}$$

We're gonna get

$$\mathbf{A}(\tau) = \frac{\mu_0}{4\pi r} \sum_a (Q_a \mathbf{v}_a) + \frac{\mu_0}{4\pi r c} \frac{\partial}{\partial \tau} \sum_a Q_a \mathbf{v}_a (\mathbf{n} \cdot \mathbf{r}_a) + \cdots. \tag{189}$$

In the first term we extract the time derivative before the sum (everything is now in the common retarded time τ), in the second term we perform the symmetrization (174) – but now the time derivative of the symmetric part will not be zero as in the stationary case:

$$\mathbf{A}(\tau) = \mathbf{A}^{(E1)} + \mathbf{A}^{(M1)} + \mathbf{A}^{(E2)} + \cdots ; \tag{190}$$

$$\mathbf{A}^{(E1)} = \frac{\mu_0}{4\pi r} \frac{\partial}{\partial \tau} \sum_a Q_a \mathbf{r}_a , \tag{191}$$

$$\mathbf{A}^{(M1)} = \frac{\mu_0}{4\pi r c} \frac{\partial}{\partial \tau} \sum_a \frac{1}{2} Q_a (\mathbf{r}_a \times \mathbf{v}_a) \times \mathbf{n} , \tag{192}$$

$$\mathbf{A}^{(E2)} = \frac{\mu_0}{8\pi r c} \frac{\partial^2}{\partial \tau^2} \sum_a Q_a \mathbf{r}_a (\mathbf{n} \cdot \mathbf{r}_a) . \tag{193}$$

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All three radiative potentials can be rewritten using the quantities introduced in the stationary expansion. In the first one there is an electric dipole moment, in the second one a magnetic dipole moment, only the third potential will have to be adjusted. From the previous we know – see (46) – that we can change the potential by the gradient of an arbitrary function

$$\mathbf{A} \to \mathbf{A} + \nabla f(\mathbf{r}) . \tag{194}$$

Volme pouze funkci vzdálenosti, pak bude transformace k novému potenciálu vypadat

$$\mathbf{A} \to \mathbf{A} + f'(r) \frac{\mathbf{r}}{r} \tag{195}$$

and the potential $\mathbf{A}^{(E2)}$ is easily converted into a form with an electric quadrupole moment

$$\mathbf{A}^{(E2)} = \frac{\mu_0}{8\pi rc} \frac{\partial^2}{\partial \tau^2} \frac{1}{3} \sum_a 3Q_a \mathbf{r}_a (\mathbf{n} \cdot \mathbf{r}_a) \;\to\; \frac{\mu_0}{24\pi rc} \frac{\partial^2}{\partial \tau^2} \sum_a Q_a \left[3\mathbf{r}_a (\mathbf{n} \cdot \mathbf{r}_a) - r_a^2 \mathbf{n} \right] . \tag{196}$$

The resulting relations for the radiative potentials are therefore (everything is expressed in the common retardation time τ, the time derivatives are also according to τ)

$$\blacktriangleright \qquad\qquad \mathbf{A}^{(E1)} = \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}_E , \tag{197}$$

$$\blacktriangleright \qquad\qquad \mathbf{A}^{(M1)} = \frac{\mu_0}{4\pi rc} \dot{\mathbf{p}}_M \times \mathbf{n} , \tag{198}$$

$$\blacktriangleright \qquad\qquad \mathbf{A}^{(E2)} = \frac{\mu_0}{24\pi rc} \ddot{\vec{\mathbf{Q}}} \cdot \vec{\mathbf{n}} . \tag{199}$$

The last expression is a contraction of the tensor $\vec{\mathbf{Q}}$ and vector $\vec{\mathbf{n}}$.

Electric dipole radiation

Le us determine the electric and magnetic fields for the potential (197). We must remember that the joint retardation time is also a function of r (observer position) and that it holds

$$\tau = t - r/c \qquad \Rightarrow \tag{200}$$

$$\frac{\partial \tau}{\partial t} = \frac{\partial t}{\partial \tau} = 1 ; \quad \frac{\partial \tau}{\partial x_k} = -\frac{1}{c}\frac{x_k}{r} . \tag{201}$$

Now we proceed to the actual calculation of the electric and magnetic fields. In the derivation we will select only radiative terms with radial dependence $1/r$, higher powers are neglected. We will denote the radiative fields constructed in this way by symbols:

$$\mathcal{E} = -\frac{\partial \mathbf{A}^{(E1)}}{\partial t} = -\frac{\partial \mathbf{A}^{(E1)}}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\mu_0}{4\pi r} \ddot{\mathbf{p}}_E ;$$

$$\mathcal{B}_k = \left(\mathrm{rot}\, \mathbf{A}^{(E1)} \right)_k = \frac{\mu_0}{4\pi} \varepsilon_{klm} \frac{\partial}{\partial x_l} \left(\frac{\dot{p}_m}{r} \right) = \frac{\mu_0}{4\pi} \varepsilon_{klm} \frac{\dfrac{\partial \dot{p}_m}{\partial \tau}\dfrac{\partial \tau}{\partial x_l} r - \dot{p}_m \dfrac{\partial r}{\partial x_l}}{r^2} .$$

The second term behaves like $1/r^2$, it is not radiative and we omit it:

$$\mathcal{B}_k = \frac{\mu_0}{4\pi r} \varepsilon_{klm} \ddot{p}_m \left(-\frac{1}{c}\frac{x_l}{r} \right) = -\frac{\mu_0}{4\pi cr} \left(\mathbf{n} \times \ddot{\mathbf{p}}_E \right)_k .$$

The result is:

$$\blacktriangleright \qquad \mathcal{E}^{(E1)} = -\frac{\mu_0}{4\pi r}\ddot{\mathbf{p}}_E \; ; \tag{202}$$

$$\blacktriangleright \qquad \mathcal{B}^{(E1)} = -\frac{\mu_0}{4\pi c r}\left(\mathbf{n}\times\ddot{\mathbf{p}}_E\right) = \frac{1}{c}\left(\mathbf{n}\times\mathcal{E}^{(E1)}\right). \tag{203}$$

It can be seen that quite far from the source the electromagnetic field has the character of a plane wave. Near the source the field would be much more complicated. The solution found is valid only in the so-called radiation zone. When calculating the field of all types of radiation, it is sufficient to calculate only the electric field in the radiation zone and to calculate the magnetic field from the relation for the plane wave (it is faster than using the rotation of the vector potential)

$$\blacktriangleright \qquad \mathcal{B} = \frac{1}{c}\left(\mathbf{n}\times\mathcal{E}\right). \tag{204}$$

For radiative fields we now determine the energy flux (Poynting vector). Only the terms $1/r^2$ remain in it:

$$\mathcal{S} = \mathcal{E}\times\mathcal{H} = \frac{1}{\mu_0 c}\mathcal{E}\times\left(\mathbf{n}\times\mathcal{E}\right) = \frac{1}{\mu_0 c}\left[\mathcal{E}^2\mathbf{n} - (\mathcal{E}\cdot\mathbf{n})\mathcal{E}\right] \tag{205}$$

We will be interested in the projection of the Poynting vector in the direction from the charges to us (radial energy flow, intensity), i. e.

$$\mathcal{S}_n = \mathcal{S}\cdot\mathbf{n} = \frac{1}{\mu_0 c}\left[\mathcal{E}^2 - (\mathcal{E}\cdot\mathbf{n})^2\right] = \frac{1}{\mu_0 c}\left[\mathcal{E}^2 - \mathcal{E}^2\cos^2\theta\right] = \frac{\mathcal{E}^2}{\mu_0 c}\sin^2\theta \, .$$

We now substitute the electric field from (202) into the radial energy flux and convert the permeability to permittivity using the relation $c^2 = 1/\varepsilon_0\mu_0$. We obtain the radial energy flux

$$\mathcal{S}_n = \frac{\ddot{p}_E^2}{16\pi^2\varepsilon_0 c^3 r^2}\sin^2\theta \, ; \qquad \theta = \sphericalangle(\mathcal{E},\mathbf{n}) = \sphericalangle(\ddot{\mathbf{p}}_E,\mathbf{n}), \qquad [\mathcal{S}_n] = \frac{\mathrm{W}}{\mathrm{m}^2}, \tag{206}$$

where θ is the angle between the far electric field (the second time derivative of $\mathbf{p}_E$) and the direction towards the observer. Therefore, the particles radiate most intensely in the direction perpendicular to the second time derivative of the electric dipole moment (for one particle, the direction perpendicular to the acceleration)
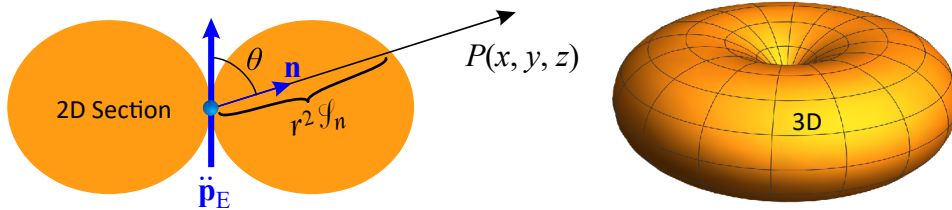
The power element radiated into the solid angle is

$$\mathrm{d}\mathcal{P} = \mathcal{S}\,\mathrm{d}\mathbf{S} = (\mathcal{S}\cdot\mathbf{n})\,\mathrm{d}S = \mathcal{S}_n\,\mathrm{d}S = \mathcal{S}_n\,r^2\,\mathrm{d}\Omega \, . \tag{207}$$

After substituting from (206) we obtain *Larmor's differential formula*

$$\blacktriangleright \qquad \mathrm{d}\mathcal{P} = \frac{\ddot{p}_E^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3}\,\mathrm{d}\Omega \, . \tag{208}$$

The total radiated power is obtained by integration

$$\mathcal{P} = \int \frac{\ddot{p}_E^2 \sin^2 \theta}{16\pi^2 \varepsilon_0 c^3} \, \mathrm{d}\Omega = \frac{\ddot{p}_E^2}{16\pi^2 \varepsilon_0 c^3} \int \sin^2 \theta \sin \theta \, \mathrm{d}\varphi \mathrm{d}\theta = \frac{\ddot{p}_E^2}{16\pi^2 \varepsilon_0 c^3} 2\pi \int_0^\pi (1 - \cos^2 \theta) \sin \theta \, \mathrm{d}\theta =$$

$$= \frac{\ddot{p}_E^2}{8\pi \varepsilon_0 c^3} \int_1^{-1} (1 - \xi^2)(-\mathrm{d}\xi) = \frac{\ddot{p}_E^2}{8\pi \varepsilon_0 c^3} \left[\xi - \frac{\xi^3}{3} \right]_{-1}^{+1} = \frac{\ddot{p}_E^2}{6\pi \varepsilon_0 c^3} \qquad \Rightarrow$$

▶
$$\mathcal{P} = \frac{\ddot{p}_E^2}{6\pi \varepsilon_0 c^3} \, ; \qquad [\mathcal{P}] = \mathrm{W}. \tag{209}$$

Several interesting consequences follow from the *Larmor formula* (209):

1) If the particles are moving with acceleration, they radiate because

$$\ddot{\mathbf{p}}_E = \frac{\partial^2}{\partial \tau^2} \sum_a Q_a \mathbf{r}_a = \sum_a Q_a \ddot{\mathbf{r}}_a \, . \tag{210}$$

2) If it is an isolated system of equal charges ($Q_a = q$) localized in one region (around the origin of our coordinates), it does not radiate, because the second time derivative of the electric dipole moment will be proportional to the sum of all internal forces, which is zero in the isolated system:

$$\ddot{\mathbf{p}}_E = \frac{\partial^2}{\partial \tau^2} \sum_a Q_a \mathbf{r}_a = \frac{q}{m} \sum_a m \ddot{\mathbf{r}}_a = \frac{q}{m} \sum_a \mathbf{F}_a = 0 \, . \tag{211}$$

3) The simplest realization of electric dipole radiation is the so-called Hertz dipole. It is an open linear element flowing with an alternating current (powered, for example, inductively from an AC circuit). Consideration 2) does not apply now, it is not an isolated system. Electric charges travel back and forth in this element, leading to a simple relation for the electric dipole moment

$$\mathbf{p}_E = p_0 \cos(\omega t) \, \mathbf{e}_3 . \tag{212}$$

If we consider that the mean value of the square of the cosine is ½, we get from the relation (209) for the mean power

$$\langle \mathcal{P} \rangle = \frac{p_0^2 \omega^4}{12\pi \varepsilon_0 c^3} . \tag{213}$$

Even a slight increase in frequency will result in a large increase in radiated power.
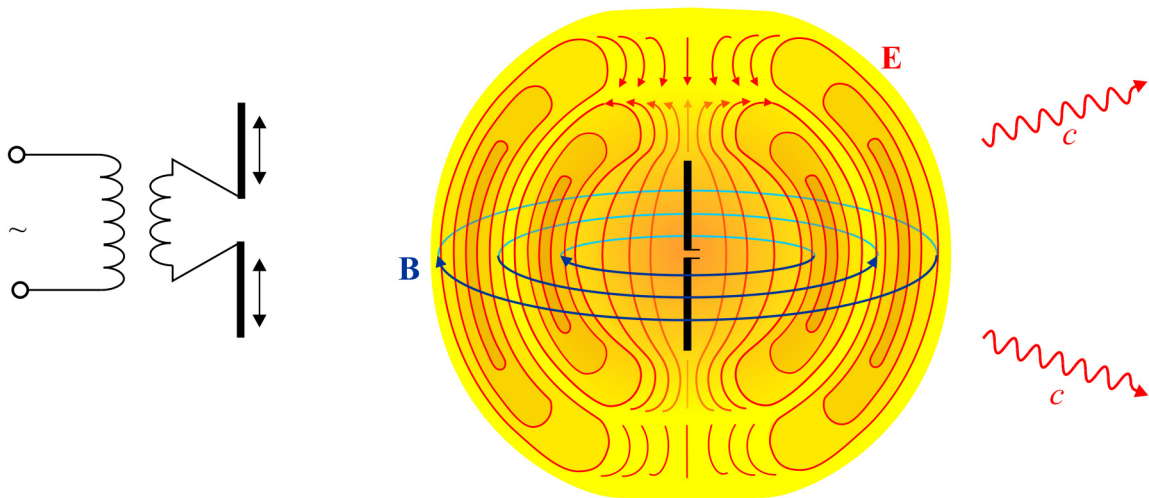
Hertz dipole. Left: simple implementation. Right: near-fields around the Hertz dipole. Two wavelengths corresponding to two flips of the dipole are shown. The relations we have derived are valid only at large distances from the dipole.

Thomson scattering

Let us now examine the radiation of a free electron on which a low-energy electromagnetic wave is incident (electron scattering on electromagnetic radiation). The electric component of the wave will exert a force $\mathbf{F} = -e\mathbf{E}$ on the electron and give it an acceleration

$$\ddot{\mathbf{r}}_e = -\frac{e}{m_e}\mathbf{E}\,. \tag{214}$$

We have neglected the effect of the magnetic field because we are in the low-energy limit where $v \ll c$. The result will be a non-zero electric dipole moment whose second time derivative will be

$$\ddot{\mathbf{p}}_E = -e\ddot{\mathbf{r}}_e = \frac{e^2}{m_e}\mathbf{E}\,. \tag{215}$$

The free electron begins to oscillate in agreement with the incoming wave, obtains a non-zero periodic acceleration and begins to radiate itself. The power emitted by it is determined from Larmor's formula (208), to which we substitute the second time derivative of the electric dipole moment from relation (215):

$$d\mathcal{P}_{out} = \frac{\ddot{p}_E^2 \sin^2\theta}{16\pi^2\varepsilon_0 c^3}\,d\Omega = \frac{e^4 E^2}{16\pi^2\varepsilon_0 c^3 m_e^2}\sin^2\theta\,d\Omega\,. \tag{216}$$

The probability of scattering will be determined by the ratio of outgoing and incoming energy. If we represent the outgoing energy by the Larmor power (216) and the incoming energy by the Poynting vector of the incident electromagnetic wave (using that $E/B = c$)

$$\mathcal{S}_{in} = EH = \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0}\,, \tag{217}$$

their ratio will be

$$\frac{d\mathcal{P}_{out}}{\mathcal{S}_{in}} = \frac{dW/dt}{dW/(dt\,dS)} = dS = d\sigma\,, \tag{218}$$

e. g. equal to an element of the area that the electron is able to receive the incident radiation, which is the effective cross section. Therefore, for the effective scattering cross section of an electron on an electromagnetic wave, we have

$$d\sigma = \frac{d\mathcal{P}_{out}}{\mathcal{S}_{in}} = \frac{\dfrac{e^4 E^2}{16\pi^2\varepsilon_0 c^3 m_e^2}\sin^2\theta\,d\Omega}{\dfrac{E^2}{c\mu_0}} = \frac{e^4 c\mu_0}{16\pi^2\varepsilon_0 c^3 m_e^2}\sin^2\theta\,d\Omega\,. \tag{219}$$

We convert the permeability to permittivity and obtain the resulting differential effective cross section

$$\blacktriangleright \qquad\qquad d\sigma = \frac{e^4}{16\pi^2\varepsilon_0^2 c^4 m_e^2}\sin^2\theta\,d\Omega\,. \tag{220}$$

The total effective cross section is obtained by integration over the spatial angle

$$\blacktriangleright \qquad\qquad \sigma_{tot} = \frac{e^4}{6\pi\varepsilon_0^2 c^4 m_e^2}\,. \tag{221}$$

The total effective scattering cross section of electromagnetic radiation on an electron is given by a universal constant, which for historical reasons is usually written by the so-called

classical radius of the electron, which we consider to be the dimension at which the electrostatic field generates the rest mass of the electron:

$$\frac{e^2}{4\pi\varepsilon a_{\mathrm{e}}} = m_{\mathrm{e}}c^2 \qquad \Rightarrow \qquad a_{\mathrm{e}} = \frac{e^2}{4\pi\varepsilon m_{\mathrm{e}}c^2}; \qquad a_{\mathrm{e}} \simeq 2.8\times10^{-15} \text{ m.} \tag{222}$$

The effective cross section in this notation comes out as a small multiple of the cross section of the electron represented by the classical radius of the electron:

$$\blacktriangleright \qquad\qquad \sigma_{\mathrm{tot}} = \frac{8}{3}\pi a_{\mathrm{e}}^2; \qquad \sigma_{\mathrm{tot}} \simeq 6.7\times10^{-29} \text{ m}^2. \tag{223}$$

Comment 1: The calculation performed is non-relativistic and valid only for low radiation energies. The frequencies of the incident wave and the wave generated by the electron are the same. Moreover, the effective cross section does not depend on the frequency. For high energies we would have to take into account the conservation of the relativistic four-momentum. A photon changes direction of motion, thus changing its momentum and of course energy (i. e., frequency), which is linked to momentum through the magnitude of the four-vector momentum. The impact of a photon with high energy will result in scattering of the photon on the electron, which will lead to a change in its frequency. In this case we speak about the Compton effect. Thomson scattering is its limit at low energies.

Comment 2: The incident wave scatters the electron simultaneously in different directions according to formula (220). If we observe one particular scattered wave in the direction θ, the plane of polarization will not change. The electric field will of course be perpendicular to the new propagation direction. The scattering angle is denoted by χ in the figure.
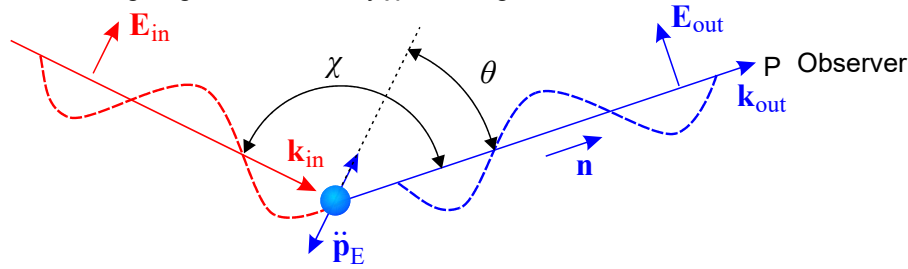
Bremsstrahlung

When an electron is in the field of a heavy ion, an elastic collision occurs, which changes the direction of motion of the electron. A centripetal acceleration (towards the ion) is applied to the electron during the collision, resulting in radiation of the electron. If we describe the collision in the coordinate system associated with the heavy ion, the magnitude of the acceleration of the electron will be

$$\ddot{r}_{\mathrm{e}} = \frac{F}{m_{\mathrm{e}}} = \frac{Ze^2}{4\pi\varepsilon_0 r_{\mathrm{e}}^2 m_{\mathrm{e}}} \tag{224}$$

and the corresponding magnitude of the second time derivative of the electric dipole moment

$$\ddot{p}_{\mathrm{E}} = e\ddot{r}_{\mathrm{e}} = \frac{Ze^3}{4\pi\varepsilon_0 r_{\mathrm{e}}^2 m_{\mathrm{e}}}. \tag{225}$$

From Larmor's formula (209) it is clear that the radiation depends only on the magnitude of the acceleration of the electron. It is quite indifferent whether the acceleration is tangential or centripetal. The magnitude of the acceleration in a Coulomb collision is determined only by the position of the electron relative to the ion at which the electron is scattered. The intensity of the radiation or the power is therefore determined only by the distance of the electron from the nucleus. The closer the electron gets to the ion, the more it radiates. We can easily determine the instantaneous power emitted by the electron:

$$\mathcal{P}(t) = \frac{\ddot{p}_{\mathrm{E}}^2}{6\pi\varepsilon_0 c^3} = \frac{Z^2 e^6}{96\pi^3\varepsilon_0^3 c^3 m_{\mathrm{e}}^2 \left[r_{\mathrm{e}}(t)\right]^4}. \tag{226}$$

It is possible to further deal with this formula. For example, from the knowledge of the electron collision path, the Fourier spectrum of the emitted power can be calculated. Another possibility is to take a homogeneous sample of electrons with different impact parameters and average the radiated power for the whole sample of electrons distributed symmetrically around the ion

$$\langle \mathcal{P} \rangle = \frac{Z^2 e^6}{96 \, \pi^3 \varepsilon_0^3 c^3 m_e^2} \int_{r_{min}}^{\infty} \frac{n_e \, 4\pi r_e^2 \, \mathrm{d}r_e}{r_e^4} = \frac{4\pi Z^2 e^6 n_e}{96 \, \pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{r_{min}}. \tag{227}$$

But this brings other problems: the first is the lower trimming of the minimum distance, which cannot be zero. The second problem is the quantum behaviour of the electron when it interacts with the electrons of the atomic envelope. If we choose the de Broglie wavelength calculated from the most probable (thermal) velocity of the electron as the lower limit of the impact parameter

$$r_{min} \approx \frac{2\pi\hbar}{\sqrt{2 m_e k_B T_e}}, \tag{228}$$

the averaged power of a set of electrons scattering on one ion will be

$$\langle \mathcal{P} \rangle \approx \frac{Z^2 e^6 n_e}{48\pi^3 \varepsilon_0^3 c^3 m_e \hbar} \left(\frac{2 k_B T_e}{m_e} \right)^{1/2}. \tag{229}$$

If the scattering takes place on ions with concentration n_i, the power emitted per unit volume (power density) will be equal to

$$\blacktriangleright \qquad \mathcal{P} = \langle \mathcal{P} \rangle n_i \approx \frac{Z^2 e^6 n_e n_i}{48\pi^3 \varepsilon_0^3 c^3 m_e \hbar} \left(\frac{2 k_B T_e}{m_e} \right)^{1/2}; \qquad [\mathcal{P}] = \frac{W}{m^3}. \tag{230}$$

Comment: The calculation performed here is non-quantum. In the quantum calculation, the so-called Gaunt factor (named after John Arthur Gaunt) appears in the resulting relation, which is weakly frequency dependent and its value is of the order of one. Thus, we do not need to consider it in the first approximation.
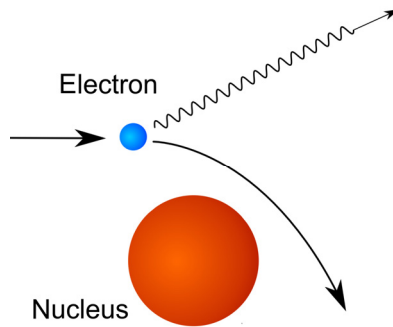
Cyclotron radiation

The charged particle performs in the magnetic field gyration motion, i. e. it moves along the helix and it has of course a non-zero centripetal acceleration given by the Lorentz force. Since cyclotron radiation is important for both electrons and ions, we perform the calculation for the general charge Q. The acceleration of the charged particle will have the magnitude:
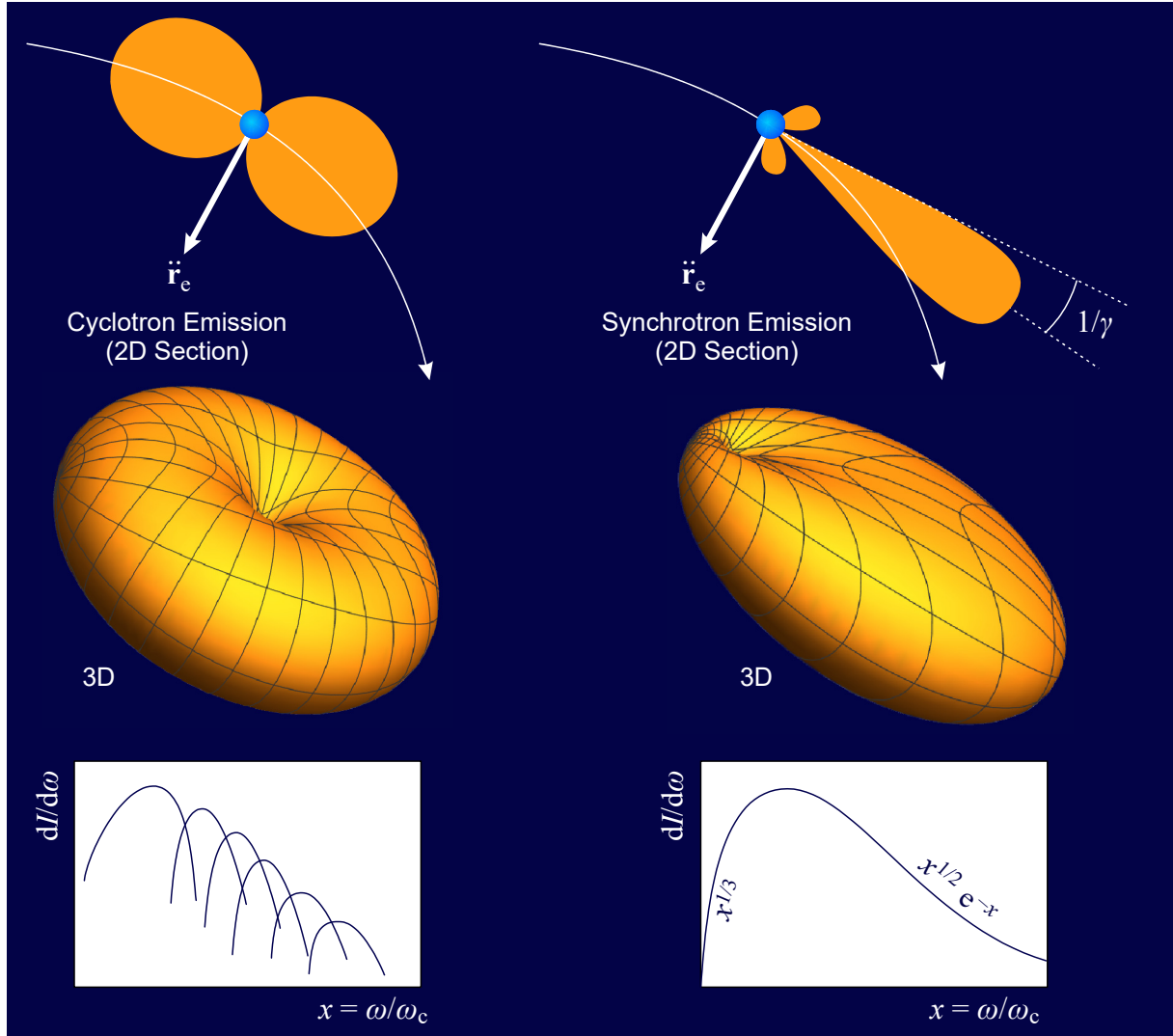
$$\ddot{r}_0 = \frac{F}{m} = \frac{Q v_\perp B}{m}. \tag{231}$$

Corresponding magnitude of the second time derivative of the electric dipole moment will be

$$\ddot{p}_E = Q \ddot{r}_0 = \frac{Q^2 v_\perp B}{m}. \tag{232}$$

From the Larmor's formula (209) we can easily determine the instantaneous power emitted by a charged particle

$$\mathcal{P} = \frac{\ddot{p}_E^2}{6\pi\varepsilon_0 c^3} = \frac{Q^4 v_\perp^2 B^2}{6\pi\varepsilon_0 c^3 m^2}. \tag{233}$$

Cyclotron Emission (2D Section)

Synchrotron Emission (2D Section)

Due to the quadratic dependence on the mass of the particle, the cyclotron emission is more intense for electrons than for ions. Specifically for the electron, using the classical radius of the electron (222), the power emitted by a single particle can be rewritten into the form

$$\mathcal{P} = \frac{8}{3}\pi a_e^2 \varepsilon_0 c v_\perp^2 B^2. \tag{234}$$

Let us calculate the energy balance for the electron component of the plasma:

$$-\frac{d}{dt}\left(\frac{3}{2}n_e k_B T_e\right) = \mathcal{P}\, n_e. \tag{235}$$

On the left is the electron energy density loss, on the right the radiated power density, in which we replace both perpendicular velocity components by the most probable (thermal) velocities

$$-\frac{d}{dt}\left(\frac{3}{2}n_e k_B T_e\right) = \frac{8}{3}\pi a_e^2 \varepsilon_0 c \frac{2k_B T_e}{m_e} B^2\, n_e. \tag{236}$$

The resulting differential equation for temperature leads to an exponential temperature drop of the electron component of the plasma due to cyclotron radiation of the electrons.

$$T_{\mathrm{e}} = T_0 \exp[-\alpha t] \, ; \qquad \alpha = \frac{32}{9} \frac{\pi a_{\mathrm{e}}^2 \varepsilon_0 c}{m_{\mathrm{e}}} B^2 . \tag{237}$$

Comment: The calculation of cyclotron radiation is non-relativistic. Cyclotron radiation takes place mainly at the cyclotron frequency and its higher harmonics. However, at high electron energies this calculation is not valid. The cyclotron frequency depends via mass on the particle velocity and, in addition, needs to be transformed into the laboratory system (the relativistic Doppler Effect applies). Similarly, the radiated power needs to be transformed. The dependence of the cyclotron frequency on the particle velocity leads for a large set of particles to a broadening of the peaks of the cyclotron emission at individual harmonics, and at high energies the spectrum will be continuous. In this case we speak of *synchrotron radiation*. The angular characteristic of the radiation is deformed by powers of the Lorentz gamma factor into the characteristic shape of a forward lobe. The formula for synchrotron radiation can be obtained by transformations of the relations for cyclotron radiation; however, a more elegant calculation is the direct calculation from Liénard-Wiechert potentials (see chapter *Radiation of an unlocalized charged particle*).

Magnetic dipole and electric quadrupole radiation

We will proceed in the same way as for electric dipole radiation. At a sufficiently large distance from the source, the wavefronts are planar, so it is enough to determine only the electric field from the relation

$$\mathcal{E} = -\frac{\partial \mathbf{A}}{\partial t} \tag{238}$$

and the magnetic field can be easily calculated from the relations

$$\mathcal{B} = \frac{1}{c} \left(\mathcal{E} \times \mathbf{n} \right) ; \tag{239}$$

$$\mathcal{E} = c \left(\mathcal{B} \times \mathbf{n} \right) . \tag{240}$$

From relations (198) and (199) we can easily determine the electric and magnetic fields of magnetic dipole and electric quadrupole radiation:

▶ $$\mathcal{E}^{(\mathrm{M1})} = -\frac{\mu_0}{4\pi rc} \ddot{\mathbf{p}}_{\mathrm{M}} \times \mathbf{n} , \tag{241}$$

▶ $$\mathcal{B}^{(\mathrm{M1})} = -\frac{\mu_0}{4\pi rc^2} \ddot{\mathbf{p}}_{\mathrm{M}} ; \tag{242}$$

▶ $$\mathcal{E}^{(\mathrm{E2})} = -\frac{\mu_0}{24\,\pi rc} \dddot{\mathbf{Q}} \cdot \vec{\mathbf{n}} , \tag{243}$$

▶ $$\mathcal{B}^{(\mathrm{E2})} = -\frac{\mu_0}{24\,\pi rc^2} \left(\left[\dddot{\mathbf{Q}} \cdot \vec{\mathbf{n}} \right] \times \mathbf{n} \right) ; \tag{244}$$

The following figure shows an example of electric quadrupole radiation in close proximity to the source, imaged using the Line Integral Convolution (LIC) method. The computer-generated noise texture is deformed in the direction of the electric field lines using electric field convolution. The calculation was performed at Duke University.
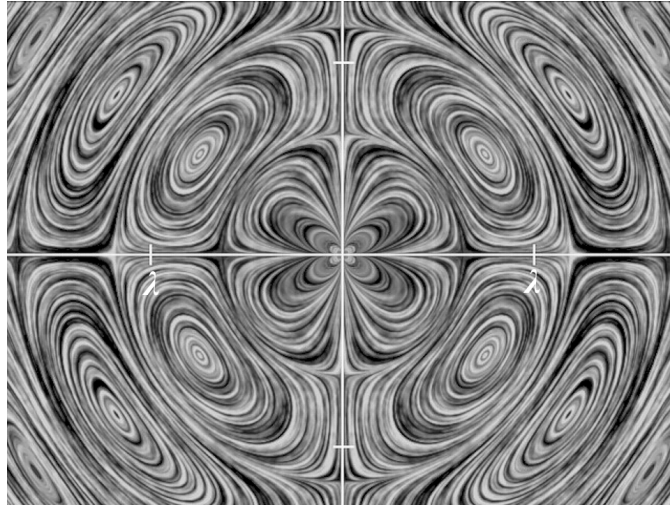
Quadrupole radiation near the source. Duke University.

Using an identical procedure as for dipole radiation, we determine the normal component of the Poynting vector and then integrate over the entire space angle to obtain the total radiated power of the particle system. In the resulting relations, we express the permeability of the vacuum in terms of permittivity from the relation $\mu_0 = 1/(c^2\varepsilon_0)$:

$$\mathscr{S}_n^{(\text{M1})} = \frac{\ddot{p}_{\text{M}}^2}{16\pi^2\varepsilon_0 c^5 r^2}\sin^2\theta ; \tag{245}$$

$$\mathscr{S}_n^{(\text{E2})} = \frac{\left(\dddot{Q}_{kl}n_l\right)\left(\dddot{Q}_{km}n_m\right) - \left(\dddot{Q}_{kl}n_k n_l\right)\left(\dddot{Q}_{mo}n_m n_o\right)}{576\,\pi^2\,\varepsilon_0 c^5\,r^2} ; \tag{246}$$

▶
$$\mathcal{P}^{(\text{M1})} = \frac{\ddot{p}_{\text{M}}^2}{6\pi\varepsilon_0 c^5} , \tag{247}$$

▶
$$\mathcal{P}^{(\text{E2})} = \frac{\dddot{Q} : \dddot{Q}}{720\,\pi\varepsilon_0 c^5} . \tag{248}$$

Radiation of an unlocalized charged particle

If the particles are moving at high velocity and are not localized around the origin of the co-ordinate system, it is not possible to perform the expansion of the potentials for an observer at large distance. In addition, the relativistic behaviour of the particles disappears with the corresponding expansion. Therefore, in the general case, a different approach is needed. Starting again from retarded potentials

$$A^\alpha = \frac{\mu_0}{4\pi}\int\frac{J^\alpha(t',\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' ; \qquad t' \equiv t - \frac{|\mathbf{r}-\mathbf{r}'(t')|}{c} . \tag{249}$$

For the scalar and vector potentials from this relation we get the formulae

$$\phi = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_Q(t',\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' , \tag{250}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{j}_Q(t',\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3\mathbf{r}' , \tag{251}$$

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For a point particle, the source terms are

$$\rho_Q = Q\,\delta(\mathbf{r'} - \mathbf{r_0}),\tag{252}$$

$$\mathbf{j}_Q = Q\,\mathbf{v}(t')\,\delta(\mathbf{r'} - \mathbf{r_0}).\tag{253}$$

For the relations (250), (251) for the potentials, it is necessary to ensure that the particle parameters are taken in retarded time, therefore for the expression for a single particle a distribution function over time is added:

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{Q\,\delta(\mathbf{r'}-\mathbf{r_0})\,\delta\left(t'-t+|\mathbf{r}-\mathbf{r'}|/c\right)}{|\mathbf{r}-\mathbf{r'}|}\,dt'\,d^3\mathbf{r'},\tag{254}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{Q\,\mathbf{v}(t')\delta(\mathbf{r'}-\mathbf{r_0})\,\delta\left(t'-t+|\mathbf{r}-\mathbf{r'}|/c\right)}{|\mathbf{r}-\mathbf{r'}|}\,dt'\,d^3\mathbf{r'}.\tag{255}$$

The time distribution will be non-zero only for

▶
$$t' = t - \frac{|\mathbf{r}-\mathbf{r'}(t')|}{c},\tag{256}$$

so actually in retarded time, as we requested. Let us first perform the integration over the spatial distribution:

$$\phi = \frac{Q}{4\pi\varepsilon_0}\int \frac{\delta\left(t'-t+|\mathbf{r}-\mathbf{r_0}|/c\right)}{|\mathbf{r}-\mathbf{r_0}|}\,dt',\tag{257}$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}Q\int \frac{\mathbf{v}(t')\delta\left(t'-t+|\mathbf{r}-\mathbf{r_0}|/c\right)}{|\mathbf{r}-\mathbf{r_0}|}\,dt'.\tag{258}$$

Integration over time is a bit more complicated, as it is a distribution function for which is valid the relation

▶
$$\delta(F(\xi)) = \frac{\delta(\xi-\xi_0)}{F'(\xi_0)};\qquad F(\xi_0)=0,\tag{259}$$

which we will now use. Let us first find the derivative of the function F:

$$\frac{dF}{dt'} = \frac{d}{dt'}\left(t'-t+|\mathbf{r}-\mathbf{r'}(t')|/c\right) =$$

$$= \frac{d}{dt'}\left(t'-t+\frac{1}{c}\sqrt{(\mathbf{r}-\mathbf{r'}(t'))^2}\right) =$$

$$= 1 + \frac{1}{c}\left(-\frac{d\mathbf{r'}}{dt'}\right)2(\mathbf{r}-\mathbf{r'}(t'))\frac{1}{2}\frac{1}{\sqrt{(\mathbf{r}-\mathbf{r'}(t'))^2}} =$$

$$= 1 - \frac{\mathbf{v}\cdot\mathbf{R}}{c\,R} = 1 - \boldsymbol{\beta}\cdot\mathbf{n}.\tag{260}$$

For a non-localized particle we have to choose the direction from the particle to us as the unit vector of the normal to the surface at the observer's location, i. e. $\mathbf{n} = (\mathbf{r} - \mathbf{r_0})/|\mathbf{r} - \mathbf{r_0}|$, hereafter we denote $\boldsymbol{\beta} = \mathbf{v}/c$.
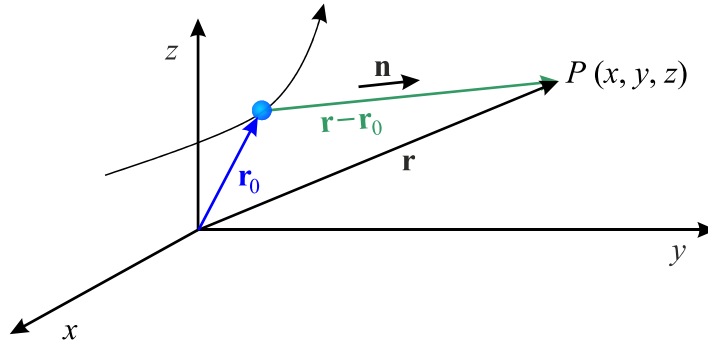
After performing the integration over the time distribution, we now easily obtain the resulting potentials (the so-called *Liénard-Wiechert potentials*):

$$\phi = \frac{Q}{4\pi\varepsilon_0 R}\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})}, \tag{261}$$

$$\mathbf{A} = \frac{Q\boldsymbol{\beta}}{4\pi\varepsilon_0 cR}\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})}, \tag{262}$$

where we denoted

$$\mathbf{n} \equiv \frac{\mathbf{R}}{R} = \frac{\mathbf{r}-\mathbf{r}_0(t')}{\left|\mathbf{r}-\mathbf{r}_0(t')\right|}. \tag{263}$$

$$\boldsymbol{\beta} \equiv \frac{\mathbf{v}(t')}{c}. \tag{264}$$

All arguments related to the particle are taken in retarded time, i. e. $\mathbf{r}_0(t')$, $\mathbf{v}(t')$. The Liénard-Wiechert potentials are relativistic and no expansions or simplifications have been made. We now determine the electromagnetic fields by direct calculation. To do this, we must precompute the various derivatives needed:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t} \tag{265}$$

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x_k} \tag{266}$$

For example, let's determine $\partial t'/\partial t$:

$$\frac{\partial t'}{\partial t}\frac{\partial t}{\partial t'} = 1; \quad \Rightarrow \quad \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\left[t'+\frac{\left|\mathbf{r}-\mathbf{r}_0(t')\right|}{c}\right] = 1; \quad \Rightarrow \quad \frac{\partial t'}{\partial t}\left[1-\frac{\mathbf{v}\cdot\mathbf{R}}{cR}\right] = 1$$

From here we can easily calculate

$$\frac{\partial t'}{\partial t} = \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}. \tag{267}$$

By analogy we find

$$\frac{\partial t'}{\partial x_k} = -\frac{n_k}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}. \tag{268}$$

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The calculation of electric fields from Liénard-Wiechert potentials is now straightforward but somewhat tedious. The electric field is obtained from the relation $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and the magnetic field from the relation $\mathbf{B} = \text{rot } \mathbf{A}$. The result is the Feynman formula

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\frac{\left(1-\beta^2\right)(\mathbf{n}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{Q}{4\pi\varepsilon_0 cR}\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3},\tag{269}$$

$$\mathbf{B} = \frac{\mathbf{n}\times\mathbf{E}}{c}.\tag{270}$$

The first part of the field is a generalized Coulombic field, the second part is a general radiative field in which no expansions have been made, and is fully relativistic. From the formulas thus determined, the Poynting vector and the power radiated by its integration can be further calculated. By such a procedure, for example, an exact relation for synchrotron radiation can be obtained.

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5. SOLITONS

From the Union Canal to solitons

In some situations, nonlinear equations provide very interesting solutions: solitary waves that are not subject to dispersion and propagate through the medium without changing shape. Such a wave is called a soliton. From a mathematical point of view, solitons in nonlinear theories are as important as the harmonic oscillator in linear theories. The soliton was first observed experimentally on a narrow water channel (the Union Canal at Hermiston, near Edinburgh) by the Scottish scientist John Scott Russell (1808–1882) in 1834.

The origins of soliton physics are linked to Russell's name once and for all. Russell was not only an outstanding physicist, but also a designer of ships, steam cars and an inventor. He was educated at the University of St Andrews and was so knowledgeable that he became a professor at the University of Edinburgh at the age of 24. Two years later, in 1834, he introduced steam car services between Glasgow and Paisley. His initial success was hardly borne by the other operators and they engineered an accident in which four passengers died. It was probably the first ever motor vehicle accident. Russell therefore welcomed the offer of the Union Canal Company to test and construct boats on the Union Canal. This very narrow canal was built between 1818 and 1822, connecting the cities of Eddinburg and Glasgow. It was primarily used for the transport of coal and people in barges, which had to be drawn by horses trotting along the canal banks. The canal is 50 kilometres long, mostly following the contour of the land and crossing rough terrain by aqueducts or tunnels (the longest tunnel is 631 metres long). In August 1834, Russell made an extraordinary discovery, which he describes as follows:

„I looked at the boat, which was being drawn rapidly along the narrow canal by a pair of horses. The boat suddenly stopped, but not so the water that was pushed in front of the bow of the boat. The water continued to move on its own, rolling down the canal at great speed. It formed into a solitary heaving wave with a smooth round shape. Neither the shape of the wave nor its speed changed as it moved. I followed the wave on horseback and overtook it. It was still rolling along at eight or nine miles an hour, not changing its shape; it was about thirty feet long and a foot and a half high. The height gradually diminished until I lost the wave in a bend of the canal after a mile or two. I first met with this extraordinary and beautiful phenomenon, which I have called the translational wave, in August, 1834.“
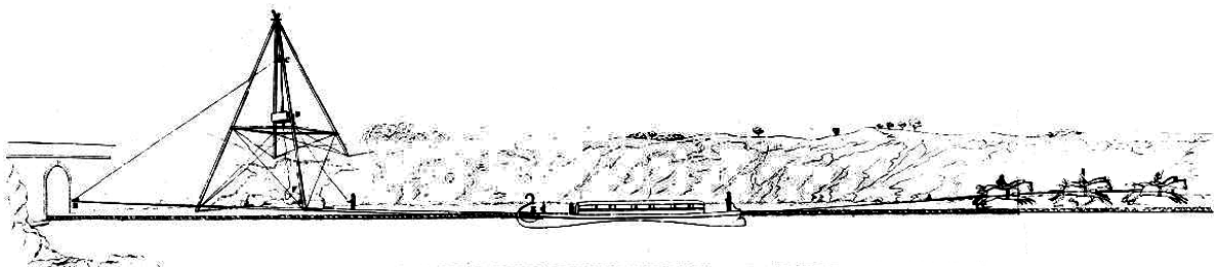
Russell's drawing of a horse-drawn barge on the Union Canal

Russell was fully aware of the uniqueness of the phenomenon. The *Great Wave of Translation* he discovered contradicted the laws of hydrodynamics described by Newton and Bernoulli in their works. Russell built a pond at his house in the New Town district of Edinburgh and for three years conducted various experiments, both at home and on the Union Canal. He found that the *translational wave* had very strange and unexpected properties:

- The speed depends on the size of the wave and the width on the depth of the water. A translational wave is extremely stable and can hold its shape for several miles.

- Unlike normal waves, translational waves never couple. A faster wave passes through a slower one and overtakes it.

If the translational wave is too large relative to the depth of the channel, it will split into two waves, one large and one small. Russell's research has not been well received by other scientists. Scholars were not prepared for such a discovery and were unwilling to believe that there were phenomena that contradicted the work of Newton and Bernoulli. The real significance was not appreciated until the 1960s, when physicists began to look at nonlinear phenomena in nature. Today, no one is surprised that light pulses travel thousands of kilometres without losing their shape when communicating over optical fibres.

In 1895, Dutch mathematicians Diederik Korteweg (1848–1941) and Gustav de Vries (1866–1934) derived an equation for wave propagation in shallow water. The equation is nonlinear and the dispersion of the wave can be exactly compensated by nonlinear phenomena. Sixty years after Russell's observations, the problem was solved. Waves can indeed propagate on a water channel without changing shape or size, and their existence can be explained theoretically using nonlinear phenomena. The equation describing this wave is now called the KdV (Korteweg-de Vries) equation. A number of other equations from different fields of science are now known to provide soliton solutions.

What is the mathematical basis for the existence of solitons? In linear equations, we can compose the resulting solution from plane wavefronts. However, the composite wavepacket is almost always subject to *dispersion*. Different wavelengths propagate at different speeds and the packet dissipates. In nonlinear theories, some nonlinear terms cause a so-called *modular instability* in which the group velocity depends on the amplitude of the packet. The parts with smaller amplitude are suppressed and the packet collapses. In some cases, the two phenomena of dispersion (dissipation) and modular instability (collapse) may compensate each other. The result is a *soliton*, a wave of stable shape and size propagating through the environment. Solitons, as solutions to nonlinear equations, are found in a variety of mathematical models in many fields, from hydrodynamics to nonlinear electrodynamics, plasma physics, biophysics, elementary particle physics, solid state physics, and economic models of market functioning.

Three examples

1. Let us first examine the dispersion relation of the ordinary wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \quad \Rightarrow \quad \omega = ck \quad \Rightarrow \quad v_f = v_g = c. \tag{271}$$

   The angular frequency is linearly dependent on the wave vector, the phase and group velocity are equal and constant. All wavelengths propagate at the same speed and the wave packet is not subject to dispersion.

2. Now consider the Klein-Gordon equation, which has an extra linear term (with a coefficient proportional to the square of the particle mass)

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2\right)\phi = 0 \quad \Rightarrow \quad \omega = c\sqrt{k^2 + \kappa^2} \quad \Rightarrow \quad \begin{aligned} v_f &= c\sqrt{1 + (\kappa/k)^2}\,, \\ v_g &= c/\sqrt{1 + (\kappa/k)^2}\,. \end{aligned} \tag{272}$$

   The phase and group velocity depend on the wave vector, i. e. also on the wavelength ($k = 2\pi/\lambda$) and the wave is subject to dispersion. The wave packet gradually dissipates. If its linear dimension is L, it varies according to the relation

$$\Delta L = \frac{\partial L}{\partial t}\Delta t \approx \Delta v_g \Delta t = \frac{\partial v_g}{\partial k}\Delta k \Delta t \approx \frac{\partial v_g}{\partial k}\frac{2\pi}{L}\Delta t\,. \tag{273}$$

3. Let us now add a non-linear term to the wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi + \gamma\phi^3 = 0. \tag{274}$$

The equation is non-linear and its solution is no longer a plane wave, nor can the solution be composed of plane waves. From the numerical solution it is known that the equation is subject to *modulation instability*, parts of the packet collapse depending on the amplitude (modulus, hence the name modulation).
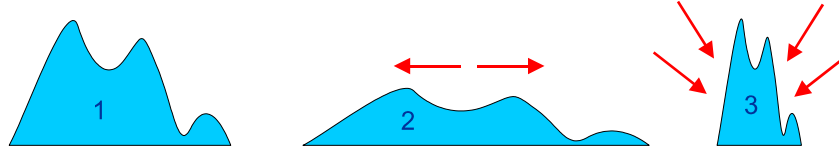
Solution shape for examples 1, 2 and 3 above

It is the nonlinearities that can balance the dispersion, and an equation that contains both terms provides soliton solutions, i. e. solitary waves propagating unchanged through a given medium:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi - \kappa^2\phi + \gamma\phi^3 = 0, \tag{275}$$

The added terms are "force" densities corresponding to the density of the "bottom of the cognac bottle" potential $\mathscr{V}(\phi) = a\phi^2 - b\phi^4$, which is widely used in elementary particle theory, in the description of phase transitions of the second order or in the description of bifurcations (branching solutions).

Soliton and soliton step wave

For a continuous environment, one should use the density of the Lagrange function $\mathscr{L}$ instead of the Lagrange function L, similarly the energy density $\mathscr{E}$ and the momentum density $\mathscr{P}$. All quantities are functions of the event $(t, \mathbf{x})$ instead of functions of time t. For our purposes, the energy density will be the most important

▶ $$\mathscr{E}(t,\mathbf{x}) \equiv \pm\left(\frac{\partial\mathscr{L}}{\partial\phi_{k,t}}\phi_{k,t} - \mathscr{L}\right). \tag{276}$$

If we do not demand anything from energy other than the law of conservation under symmetry with respect to time displacement, the sign is irrelevant. However, if we want the forces to be directed towards the minimum of the energy, and for compact objects this definition translates into the classical definition of energy, we must choose the sign so that the kinetic part (the square of the derivative of the field) is non-negative (the result will be affected by the sign convention used in the metric tensor, in our case we must choose minus for the energy). For the examples from the last chapter (in one spatial dimension for simplicity), we can write for the Lagrangian density, the energy density, and the Lagrange field equation:

Wave equation

$$\mathscr{L} = \frac{1}{2}\left(\partial_\alpha\phi\right)\left(\partial^\alpha\phi\right) = -\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2, \tag{277}$$

$$\mathscr{E} = -\left(\frac{\partial\mathscr{L}}{\partial\phi_{k,t}}\phi_{k,t} - \mathscr{L}\right) = \frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2, \tag{278}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0. \tag{279}$$

Klein-Gordon equation

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha \phi\right)\left(\partial^\alpha \phi\right) + \frac{1}{2}\kappa^2 \phi^2, \tag{280}$$

$$\mathcal{E} = -\left(\frac{\partial \mathcal{L}}{\partial \phi_{k,t}}\phi_{k,t} - \mathcal{L}\right) = \frac{1}{2c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\kappa^2 \phi^2, \tag{281}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2\right)\phi = 0. \tag{282}$$

Kleinova-Gordon equation with nonlinear term

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha \phi\right)\left(\partial^\alpha \phi\right) + \frac{1}{2}\kappa^2 \phi^2 - \frac{1}{4}\delta \phi^4, \tag{283}$$

$$\mathcal{E} = -\left(\frac{\partial \mathcal{L}}{\partial \phi_{k,t}}\phi_{k,t} - \mathcal{L}\right) = \frac{1}{2c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\kappa^2 \phi^2 - \frac{1}{4}\delta \phi^4, \tag{284}$$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2\right)\phi + \delta \phi^3 = 0. \tag{285}$$

For the Klein-Gordon equation with a nonlinear term ϕ^3, a solution in the form of a propagating step structure is known. It is not a soliton in the strict sense of the word, the solution is not localized, i. e. quite far from the wave the field is not zero. However, the course of the energy density is already localized. The energy density is concentrated in the step region and moves as a packet through space. This type of solution is called a *soliton step wave*. A solution that is truly localized in space, moves at some speed, does not change its shape, and at most undergoes a phase shift when it collides with other similar solutions, is called a *soliton*. A soliton has, of course, a localized energy density.
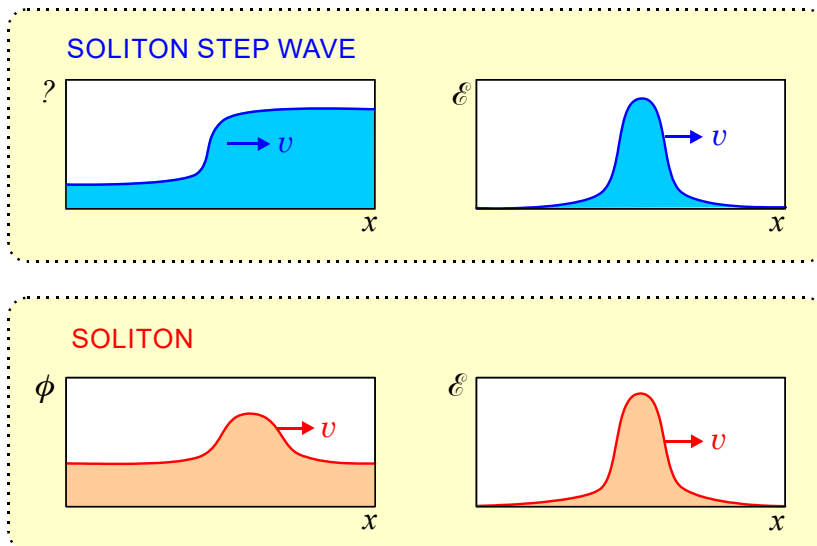
Definition: A soliton is a localized solution of a partial differential equation that moves at some speed and does not change its shape. If, before the collision of N solitons, it has an energy density

$$\blacktriangleright \qquad \mathcal{E}(t,x) = \sum_{k=1}^{N} \mathcal{E}_0(x - x_{0k} - vt), \tag{286}$$

then neither the number of solitons nor their shape changes after the collision. Only the phase of the individual packets can change, i. e. after collision it must be valid:

$$\blacktriangleright \qquad \mathscr{E}_{\text{fin}}(t,x) = \sum_{k=1}^{N} \mathscr{E}_0(x - x_{0k} - vt + \delta_k). \qquad (287)$$

When looking for a soliton solution, we can take advantage of the fact that a soliton does not change its shape when it moves. In the coordinate system associated with the soliton, it is therefore a stationary (time-invariant) solution. Moreover, if we observe motion in only one spatial dimension, the partial differential equation becomes an ordinary differential equation (does not contain time), which may be solvable in some cases. After finding the solution in the system moving with the soliton, we have to transform the solution into the coordinate system in which the soliton moves.

Sin-Gordon soliton step wave

In this and the following chapters we will show typical soliton solutions to which physical problems lead. In the treatment of partial differential equations, in many cases we arrive at one of four typical equations: the sin-Gordon equation, the KdV, the NLS, or the Burgers equation. Let us now consider the first of these.

The Klein-Gordon equation with nonlinear terms (283) to (285) has a potential part in both the energy and Lagrange functions

$$\mathscr{V} = \frac{1}{2}\kappa^2\phi^2 - \frac{1}{4}\delta\phi^4. \qquad (288)$$

The first potential energy term leads to the dispersion, the second to a nonlinear term providing modular instability and thus dispersion compensation. If we look at the potential energy density more closely, we find that except for the constants, these are the first two non-zero terms of the Taylor series of the cosine function (the zero constant term is of course irrelevant and does not appear in the equations of motion). If we consider that, due to the sign convention used, we use the density of the Lagrange function in the form

$$\mathscr{L} = -\mathscr{T} + \mathscr{V}, \qquad (289)$$

we can introduce a periodic potential

$$\mathscr{V} = -\kappa^2 \cos\phi, \qquad (290)$$

which will be useful for describing phenomena in the vicinity of periodic structures and for small values of the ϕ field it will be compatible with the potential (288) of the nonlinear Klein-Gordon equation. We can easily determine the corresponding Lagrangian density, energy density and field Lagrange equation:

$$\mathscr{L} = \frac{1}{2}(\partial_\alpha\phi)(\partial^\alpha\phi) - \kappa^2 \cos\phi, \qquad (291)$$

$$\mathscr{E} = -\left(\frac{\partial\mathscr{L}}{\partial\phi_{k,t}}\phi_{k,t} - \mathscr{L}\right) = -\frac{1}{2}(\partial_\alpha\phi)(\partial^\alpha\phi) - \kappa^2 \cos\phi, \qquad (292)$$

$$\Box\phi - \kappa^2 \sin\phi = 0. \qquad (293)$$

In the field equation, the second term is preceded by a minus from the Lagrange function, a minus from the derivative of the cosine and a minus from the Lagrange equations – the result is therefore a minus. The sine in the field equation gave it its name: the *sin-Gordon equation*. Let's write down the last relations for one spatial dimension:

$$\mathcal{L} = -\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \kappa^2\cos\phi, \tag{294}$$

$$\mathcal{E} = +\frac{1}{2c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \kappa^2\cos\phi, \tag{295}$$

▶
$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \kappa^2\sin\phi. \tag{296}$$

In mechanics, the motion of a particle in a periodic cosine potential leads to an analogous problem; the force on the right side of the equation of motion is given by the sine of the particle's position. If we expand the right-hand side to first order ($\sin\phi \sim \phi$), we obtain exactly the Klein-Gordon equation. Therefore, the coefficient on the right-hand side is denoted by κ^2. The expansion to third order ($\sin\phi \sim \phi - \phi^3/3!$) gives the Klein-Gordon equation with the nonlinear term ϕ^3, which corresponds to the potential of the cognac bottle. We will now look for the solution of the full sin-Gordon equation with the sine function on the right-hand side. We will show that we can find a solution in the form of a soliton step wave. Let us first find the solution in the coordinate system associated with the soliton step wave. Since the wave does not change shape and moves at a constant speed, the solution will be stationary and it is sufficient to solve the ordinary differential equation

▶
$$\frac{d^2\phi}{dx^2} = \kappa^2\sin\phi. \tag{297}$$

Let's show that one of the solutions is

▶
$$\phi_\pm(x) = 4\,\mathrm{atg}\left[e^{\pm\kappa x}\right]. \tag{298}$$

It is a very nice logical exercise to prove that the function is a solution of the sin-Gordon equation. We can either directly differentiate the given solution twice or differentiate the resulting relation $\mathrm{tg}(\phi/4) = \exp(\pm\kappa x)$, calculate the first derivative and differentiate it again. Wherever the expression $\exp(\pm\kappa x)$ appears, we replace it with the relation $\mathrm{tg}(\phi/4)$. For the sake of clarity, we will use the solution with a "+" sign, and we would proceed similarly for the second sign:

$$\mathrm{tg}(\phi/4) = \exp(\kappa x) \qquad \left| \, d/dx,\right.$$

$$\frac{1}{4}\frac{d\phi}{dx}\frac{1}{\cos^2(\phi/4)} = \kappa\exp(\kappa x) \qquad \left|\, \exp(\kappa x) = \mathrm{tg}(\phi/4),\right.$$

$$\frac{d\phi}{dx} = 4\kappa\cos^2(\phi/4)\,\mathrm{tg}(\phi/4) \qquad \left|\, \mathrm{tg}(\phi/4) = \sin(\phi/4)/\cos(\phi/4),\right.$$

$$\frac{d\phi}{dx} = 2\kappa\sin(\phi/2) \qquad \left|\, d/dx,\right.$$

$$\frac{d^2\phi}{dx^2} = \kappa\frac{d\phi}{dx}\cos(\phi/2) \qquad \left|\, \frac{d\phi}{dx} = 2\kappa\sin(\phi/2),\right.$$

$$\frac{d^2\phi}{dx^2} = \kappa^2\sin\phi.$$

The above relation is therefore a solution of the original equation. We now transform it from the system associated with the soliton to the general laboratory system. The whole equation is relativistically covariant, so we can use the Lorentz transformation

$$x_{sol} = \frac{x_{lab} - x_0 - vt}{\sqrt{1 - v^2/c^2}}.$$
(299)

Therefore, in a laboratory system, a moving soliton step wave will have the shape

▶
$$\phi_{\pm}(t,x) = 4 \, atg\left[e^{[\pm \kappa \gamma (x - x_0 - vt)]} \right]; \qquad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}.$$
(300)

If the reader is not completely exhausted by the derivation, he can try to prove that the expression found solves the original sin-Gordon equation (296). It remains to find the energy density according to relation (295):

▶
$$\mathcal{E}_{\pm}(t,x) = \frac{8\kappa^2 \gamma^2 (1 + v^2/c^2) \, e^{[\pm 2\kappa \gamma (x - x_0 - vt)]}}{\{1 + \exp[\pm 2\kappa \gamma (x - x_0 - vt)]\}^2} - \kappa^2 \cos\left\{ 4 \, atg\left[e^{[\pm \kappa \gamma (x - x_0 - vt)]} \right] \right\}.$$
(301)

The solution is a moving step, so in terms of classification it is not a soliton, but a soliton step wave, in which the field solution is non-local, but the energy is localized at the step. The amplitude of the soliton wave does not depend on the velocity or any other parameters of the calculation. This is not the case for the energy; its amplitude is of course dependent on the soliton wave velocity. (Imagine a tsunami wave; its destructive energy undoubtedly depends on its speed). In the following figure the solutions $\phi_{\pm}$, $\mathcal{E}_{\pm}$ are plotted:
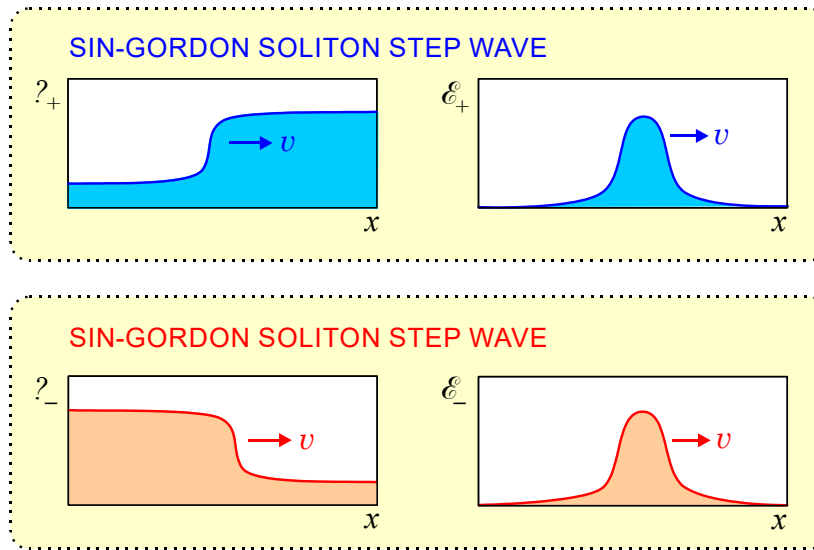

Let us now show what phenomena lead to the sin-Gordon soliton wave. There are a number of them, and we will briefly mention falling dominoes, the electric double layer, dislocation in crystals, the Josephson Effect, and the propagation of a laser pulse through a medium with two electron energy levels.

Falling dominoes. Everyone's probably seen these videos. The dominoes are stacked on the table on the narrower edge. Then someone pushes them. One by one, the dominoes fall, creating an interesting phenomenon. It looks like something is running through the row of dice, knocking them down. If we take as our field the height of the top edge of the dice above the table, i. e. $\phi = h(t, x)$, where x is the horizontal position of the bottom edge of the dice, their controlled fall will look like the ϕ_- solution in the figure. The Sin-Gordon equation is a good choice to describe the fall of the cubes. The periodicity of the potential reflects the periodic

distribution of the cubes, and the running "something" along the row of cubes is in fact the localized energy $\mathcal{E}_-$, which can be thought of as a quasiparticle (a moving packet of energy). Note that the amplitude of the soliton wave cannot be affected; it is determined only by the height of the dominoes and does not depend on the displacement rate of the region of the fall.

Electric double layer. In plasmas, there are often moving regions in which the electric potential changes from one value to another. In such a layer there is a non-zero electric field that redistributes the charges of the surrounding plasma into a characteristic electric double layer with a positive charge on one side of the layer and a negative charge on the opposite side (see textbook [1] for details). The electric double layer is in fact nothing more than a sin-Gordon soliton step wave of electric potential. The energy localized in the double layer (it must be larger than the average thermal energy of the surroundings) can accelerate the surrounding particles even to considerable energies. Electric double layers are encountered in both laboratory and space plasmas.

Dislocation in crystals. Imagine a crystal composed of two chains of atoms, one is fixed (bottom) and the other (top) can move in the periodic field of the first chain with potential $V(x) = A[1-\cos(2\pi x/a)]$, a is a lattice constant. The problem leads to the sin-Gordon equation for the shift from the equilibrium position. The solution (300) corresponds to a dislocation moving along the crystal. The solution gradually changes from zero shift from equilibrium to a shift of just one lattice constant a. The energy of the dislocation is localized in the region. Again, this is a classical example of a sin-Gordon soliton step wave.
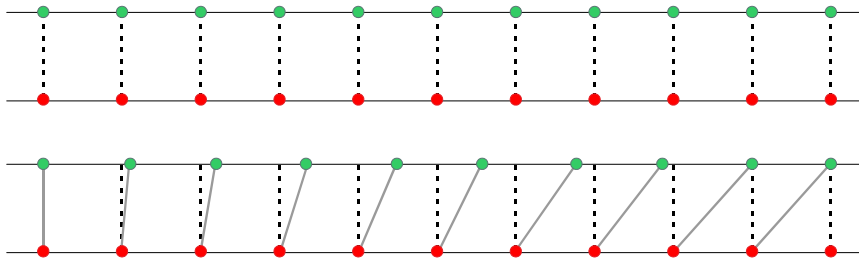
Josephson Phenomenon. As electric current passes through the thin layer separating the two superconductors, Cooper pairs tunnel through the insulator. An electric current flows through the interface, the magnitude of which depends on the external magnetic field and temperature, which can be used, for example, for SQUID (*Superconducting Quantum Interference Device*) magnetic field sensors. The magnetic induction flux passes through the Josephson junction in quanta, which are quasi-particles called *magnons*. And just the passage of the magnetic induction flux ϕ from one side of the junction to the other side is governed by the sin-Gordon equation and has the character of a sin-Gordon soliton wave. The magnon is a quasi-particle describing the motion of an energy packet localized at the "step" of the magnetic field flux.

Laser pulse propagation through a two-level environment. Let us imagine an environment made up of atoms that have two energy levels. If we send a laser pulse with a frequency corresponding to the difference of the two levels into this environment, the intensity of the laser pulse is again described by the sin-Gordon equation and has the character of a soliton wave. The ϕ_+ solution corresponds to stimulated emission at inverse level occupation and describes the amplification of the pulse in such an environment, while the ϕ_- solution describes resonant absorption of light and corresponds to the attenuation of the laser pulse.

<p align="center">* * *</p>

KdV a NLS solitons

Korteweg-de Vries soliton

The most common types of solitons include the KdV soliton satisfying the Korteweg de Vries equation and the NLS soliton satisfying the nonlinear Schrödinger equation. The KdV equation is the first equation for which a soliton solution was found. Diederik Korteweg (1848–1941) and Gustav de Vries (1866–1934) derived the equation in the hydrodynamic description of waves in shallow water. Its simplest form is as follows (we will show a simple derivation of the equation at the end of this chapter)

$$\blacktriangleright \qquad \frac{\partial \phi}{c\,\partial t} + \alpha\phi\frac{\partial \phi}{\partial x} + \delta\frac{\partial^3 \phi}{\partial x^3} = 0 \,. \tag{302}$$

The constant c is a material constant that determines the phase velocity of the propagation of the disturbances in a given medium. The constant α controls the magnitude of the nonlinear term, which is similar to the substantive derivative of the velocity field and is indeed related to the displacement of the fluid being described. The last constant δ controls the magnitude of the dispersion term, which is proportional to the third derivative of the field of interest. A soliton solution exists for any values of these constants. One can always find a wave of a velocity and height for which the dispersion is compensated by a nonlinear compression of the wave. After multiplying by the constant c, we obtain only two independent constants of the KdV equation

$$\blacktriangleright \qquad \frac{\partial \phi}{\partial t} + A\phi\frac{\partial \phi}{\partial x} + D\frac{\partial^3 \phi}{\partial x^3} = 0 \,; \tag{303}$$

$$A \equiv c\alpha \,; \qquad D \equiv c\delta \,. \tag{304}$$

Now the dispersion is controlled by the constant D and the nonlinear phenomena by the constant A. By suitable substitution for x and ϕ, we could "absorb" these constants into the variables x and ϕ. However, let us keep the KdV equation in the general form (303), from which the meaning of the constants A and D is well seen. The solution will have two basic properties:

$$\lim_{x\to\pm\infty} \phi = 0 \,, \tag{305}$$

$$\phi = \phi(\eta) \,; \qquad \eta \equiv x - x_0 \pm vt \,. \tag{306}$$

The first property of (305) expresses the locality of the solution, i. e., quite far from the soliton the function ϕ is zero. The second property (306) describes the propagation of the wave to the right (−) or to the left (+) with no change in shape. For certainty, we will look for solutions moving to the right, i. e., with a minus sign. Substituting (306) into (303), we obtain the ordinary differential equation

$$-v\frac{d\phi}{d\eta} + A\phi\frac{d\phi}{d\eta} + D\frac{d^3\phi}{d\eta^3} = 0 \,. \tag{307}$$

We adjust the middle term according to the relation $\phi\phi' = (\phi^2/2)'$ and integrate the whole equation over the variable η. We obtain an ordinary differential equation of second order

$$-v\phi + \frac{A}{2}\phi^2 + D\frac{d^2\phi}{d\eta^2} = C_1 \,. \tag{308}$$

The integration constant C_1 is zero due to the locality of the solution (305). We multiply the resulting equation by the first derivative of the function ϕ

$$-v\phi\,d\phi + \frac{A}{2}\phi^2 d\phi + D\frac{d^2\phi}{d\eta^2}d\phi = 0 \tag{309}$$

53

and the last term is modified as follows (the comma denotes the derivative by η):

$$\frac{d^2\phi}{d\eta^2}d\phi = \frac{d^2\phi}{d\eta^2}\frac{d\phi}{d\eta}d\eta = \phi''\phi'd\eta = \frac{1}{2}\frac{d(\phi')^2}{d\eta}d\eta = \frac{1}{2}d(\phi')^2. \tag{310}$$

After substituting into equation (309) and integrating, we get

$$-v\frac{\phi^2}{2} + \frac{A}{6}\phi^3 + \frac{D}{2}(\phi')^2 = C_2 \tag{311}$$

Since we are looking for a localized solution, the second integration constant is again zero and the KdV equation has switched to a first order ordinary differential equation

$$\left(\frac{d\phi}{d\eta}\right)^2 = \frac{1}{D}\phi^2\left(v - \frac{A}{3}\phi\right), \tag{312}$$

which is easy to separate. Once separated, a direct integration of this equation can be performed using suitable substitutions. The result is the solution

$$\phi(\eta) = \frac{3v}{A}\operatorname{ch}^{-2}\left[\sqrt{\frac{v}{4D}}\ \eta\right] \tag{313}$$

Indeed, for arbitrary constants there is a solution of the KdV equation in the form of a soliton:

▶ $$\phi(t,x) = \frac{3v}{A}\operatorname{ch}^{-2}\left[\sqrt{\frac{v}{4D}}\ (x - x_0 - vt)\right]. \tag{314}$$

The solution is a "hill" similar to the section of Mount Rip moving to the right with velocity v. The height of the soliton is related to the velocity of its movement:

▶ $$\phi_{\max} = \frac{3v}{A}. \tag{315}$$

Different high solitons move at different speeds, which is typical for a KdV soliton. The last relation could already be obtained from Eq. (312). This is because at maximum height the first derivative (left-hand side) is zero, and from the right-hand side we immediately have the relation we are looking for.

Comment 1: KdV equation is often given in dimensionless form with constants included in the variables. In addition, the derivatives are written as indices so that the reader can encounter the form:

$$\phi_t + 6\phi\phi_x + \phi_{xxx} = 0. \tag{316}$$

The coefficient 6 is deliberately chosen to make the solution as simple as possible. In our solution for the SI system it is sufficient to put $A = 6$, $D = 1$ and we have the solution of the KdV equation:

$$\phi(t,x) = \frac{v}{2}\operatorname{ch}^{-2}\left[\frac{\sqrt{v}}{2}\ (x - x_0 - vt)\right]. \tag{317}$$

Comment 2: The KdV equation is the simplest generalization of a wave equation of the type

$$\frac{\partial\phi}{c\partial t} + \frac{\partial\phi}{\partial x} = 0. \tag{318}$$

This simple wave equation leads to a dispersion relation

$$\omega = ck. \tag{319}$$

Let us now admit that the dispersion is not this simple and find the first simplest generalization:

$$\omega = c(k)k \approx (c_0 - c_1 k^2)k = c_0 k - c_1 k^3. \tag{320}$$

The second expansion term leads to energy dissipation (complex k), so it must be zero for soliton solutions. The third expansion term must be negative; this is the only way to have a realistic solution. If we admit as the simplest generalization of the dispersion relation the form (320), we have to modify the wave equation into the form

$$\frac{\partial \phi}{c_0 \partial t} + \frac{\partial \phi}{\partial x} + c_1 \frac{\partial^3 \phi}{\partial x^3} = 0 \,. \tag{321}$$

This wave equation can easily be rewritten in the form of a continuity equation:

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left[c_0 \phi + c_0 c_1 \frac{\partial^2 \phi}{\partial x^2} \right] = 0 \,. \tag{322}$$

The expression in square brackets represents the flux of ϕ. Its first straightforward generalization leading to nonlinear phenomena is

$$j = c_0 \phi + \alpha \phi^2 + c_0 c_1 \frac{\partial^2 \phi}{\partial x^2} \,, \tag{323}$$

which leads to a partial differential equation

$$\frac{\partial \phi}{\partial t} + c_0 \frac{\partial \phi}{\partial x} + 2\alpha \phi \frac{\partial \phi}{\partial x} + c_0 c_1 \frac{\partial^3 \phi}{\partial x^3} = 0 \,. \tag{324}$$

If we shift the independent variable $x \rightarrow x - c_0 t$, we eliminate the second term and obtain from the generalized wave equation the KdV equation of the form (303):

$$\frac{\partial \phi}{\partial t} + A\phi\frac{\partial \phi}{\partial x} + D\frac{\partial^3 \phi}{\partial x^3} = 0 \,; \qquad A = 2\alpha \,; \quad D = c_0 c_1 \,. \tag{325}$$

* * *

The KdV equation was originally derived for waves in shallow water and became the first theory to explain the observation of a soliton on a canal near Edinburgh (John Scott Russel, 1834). The solution is a typical solitary step wave that does not interact with other similar waves and whose speed depends on the wave height.

NLS soliton (Non Linear Schrödinger soliton)

A number of problems from quantum theory, but also from other fields of physics (plasma physics, Langmuir oscillations, nonlinear optics) lead to the nonlinear Schrödinger equation, which we will present only in dimensionless form:

▶
$$\mathrm{i}\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} + \sigma_{\mathrm{D}}\frac{\partial^2 \phi}{\partial y^2} + 2\sigma_{\mathrm{N}}\phi^2\phi^* = 0. \tag{326}$$

The coefficient σ_{D} determines the type of NLS dispersion; according to the value we divide NLS solitons into three cases:

- $\sigma_{\mathrm{D}} = +1$: elliptical NLS,
- $\sigma_{\mathrm{D}} = -1$: hyperbolic NLS,
- $\sigma_{\mathrm{D}} = 0$: (1+1)D NLS.

The last term represents the nonlinearity, in this case cubic, but it can be more complex. The value of the coefficient σ_{N} determines the type of modulation instability:

- $\sigma_{\mathrm{N}} = +1$: focusing,
- $\sigma_{\mathrm{N}} = -1$: de focusing.

Let us give a solution for a focusing (1+1)D NLS, which produces a so-called Davydov soliton of the form

$$\blacktriangleright \qquad \phi(t,x) \;=\; \sqrt{\omega}\, \mathrm{ch}^{-1}\!\left[\sqrt{\omega}\,(x-x_0-vt)\right] \exp\!\left[\,\mathrm{i}\frac{vx}{2} - \mathrm{i}\left(\frac{v}{2}\right)^{2} t \;+\; \mathrm{i}\,\omega t + \mathrm{i}\,\delta\right]. \qquad (327)$$

In the solution, the free parameter ω appears – the NLS soliton has a part resembling a plane wave, which is modulated (multiplied) by the envelope ch^{-1}. The amplitude is not related to the soliton velocity, but to the frequency of the wave "trapped" in the soliton. Let us compare the two basic types of solitons:

- A KdV soliton is just a bulge of the required function at the location of the wave (e. g. a water surface elevation); an NLS soliton can look like a wave packet, i. e. it can be an envelope of a wave of a certain frequency.

- The velocity of the KdV soliton depends on the amplitude, the velocity of the NLS soliton does not.

- For the KdV soliton the product of the wave height and the square of the width is constant, for the NLS soliton no such rule applies.

- KdV solitons have exactly the same shape after the collision as before the collision, at most they are phase shifted. NLS solitons retain their identity after the collision, but their shape does change somewhat.
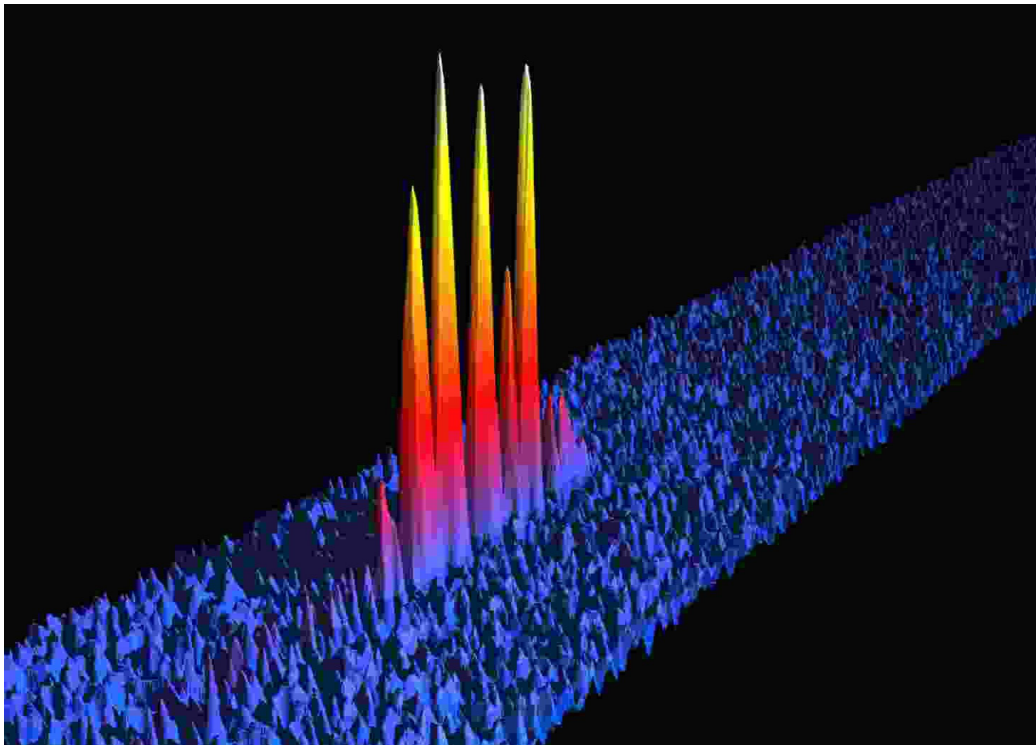
Soliton train. In 2002, scientists at Rice University succeeded in preparing a Bose-Einstein condensate with very interesting properties. The ultracold atoms formed a condensate only in certain places. A series of several moving solitons resembling a train originated. Areas of condensate moved without changing shape, as is typical of solitons.

Finally, let us show how we can obtain an equation that leads to soliton solutions in probably the simplest way. Let's modify the 1D Navier-Stokes equation

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = \eta \, \Delta u \tag{328}$$

into a simple shape

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\eta}{\rho}\frac{\partial^2 u}{\partial x^2}. \tag{329}$$

Typically, this is the so-called *Burgers equation*

▶ $$\frac{\partial u}{\partial t} + A\frac{\partial}{\partial x}\left(u^2\right) - D\frac{\partial^2 u}{\partial x^2} = 0 \tag{330}$$

with a nonlinear and dispersion term. The Burgers equation again provides soliton solutions. The procedure to find them is quite analogous to how we solved the KdV equation, i.e., we first replace the variables t, x by $\eta = x - vt$, thus converting the equation into an ordinary differential equation that is easy to integrate. The remaining first order equation is then solved by separation.

Sometimes solitons form a group moving through the environment in formation, then we speak of a *soliton train*. In some cases, the soliton seems to breathe: it periodically changes its shape and periodically returns to its initial shape. This is not a soliton in the true sense of the word (it changes shape), but a very related phenomenon to solitons, known as *breathing*.

Solitons in plasmas

Langmuir soliton. The MHD equations can be rewritten under certain assumptions to both the KdV equation and the NLS equation. An interesting phenomenon is the nonlinear modification of the plasma (Langmuir) oscillations at the plasma frequency of electrons that localize in a region isolated from the surroundings. This creates a density cavity filled with a high frequency field in which the plasma concentration is reduced according to the relation

$$\frac{\delta n}{n} \approx \frac{p_E}{p_T} \approx \frac{\varepsilon E^2/2}{2nk_BT}. \tag{331}$$

That is why the Langmuir soliton is sometimes called a well. The electric field (envelope) itself satisfies the relation

$$E(x) \approx E_0 \, \mathrm{ch}^{-1}(k_0 x)$$

and the energy density is the KdV soliton:

$$\mathcal{E}(x) = \varepsilon E^2/2 \approx E_0^2 \, \mathrm{ch}^{-2}(k_0 x)$$

However, this is only a limiting simple case. Langmuir solitons can be quite complex and behave in certain situations as NLS solitons. In fact, a Langmuir soliton can oscillate and generate low-frequency ion-acoustic waves. On the contrary, a Langmuir soliton can suck energy from ion-acoustic waves at long wavelengths by means of the so-called parametric instability. At short wavelengths, it loses energy through Landau damping. This can lead to a stationary flow of energy in k-space. A soliton interacting energetically with its surroundings is called a dissipative soliton. Langmuir solitons are usually formed by the decay of Langmuir (plasma) oscillations, hence their name. Langmuir solitons can be artificially generated in the laboratory, for example on the *Double Plasma apparatus* developed in the 1970s (see description below).

Davydov soliton. This solution is again related to Langmuir waves and is described by the Sakharov-Kuznetsov system of partial differential equations:

▶
$$i\frac{\partial \phi}{\partial t} + \frac{\partial^2 \phi}{\partial x^2} = \phi\psi;$$
$$\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2(\phi^2)}{\partial x^2}.$$
(332)

Problems in chemistry and quantum theory lead to a similar system. In plasma physics this system is suitable for the description of instabilities in the electron beam or the description of Langmuir waves in plasmas. The quantity ϕ corresponds to the electric field and the quantity ψ to the variation of the ion concentration from the equilibrium position. The electric field satisfies the Schrödinger equation with a nonlinear source term given by the perturbation of the ion concentration. The latter in turn satisfies the wave equation with a ponderomotive source term (the second derivative of the square of the electric field, i. e. the electric pressure) on the right-hand side. The system solution is called Davydov soliton:

$$\phi(t,x) = \sqrt{2(1-v^2)\omega}\ \text{ch}^{-1}\left[\sqrt{\omega}(x-vt)\right]\exp\left[i\frac{vx}{2} - i\left(\frac{v}{2}\right)^2 t + i\,\omega t + i\,\delta\right];$$
$$\psi(t,x) = -2\omega\,\text{ch}^{-2}\left[\sqrt{\omega}(x-vt)\right].$$
(333)

The solution parameters are the soliton velocity v, the phase shift δ and the scaling parameter ω. Under certain assumptions, the Sakharov-Kuznetsov equations can be simplified to either the NLS equation or the KdV equation with Langmuir soliton solution.

Trivelpiec-Gould soliton. This soliton is related to electron plasma waves in a finite space. It arises as a compression of the plasma propagating according to the KdV soliton rules. These solitons are most easily generated in a Q apparatus, which is a cylindrical tube with planar metal plates at each end that are covered with an alkali metal (cesium or potassium). The plates are heated to a high temperature and ions and electrons are evaporated into the chamber. The tube is in a strong magnetic field which reduces the diffusion of electrons perpendicular to the axis of the tube. The plasma has an equilibrium temperature of electrons and ions, which is a certain disadvantage of this arrangement, because the phase velocity of the ion-acoustic wave is comparable to the thermal velocity of the ions, and the ions therefore readily surf the ion-acoustic wave and remove its energy by Landau damping. Ideally, if the ions have a temperature much lower than the electrons, then the Landau damping on the ions is negligible. The soliton can be built up in the tube by, for example, a voltage pulse.

It has been verified experimentally that the product of the height and the square of the width is constant and that the soliton velocity depends only on its amplitude. A slight ripple of the plasma was observed behind the soliton. These waves were first described in 1959 by Alvin W. Trivelpiece and Roy W. Gould of the California Institute of Technology.

Ion-acoustic soliton. A soliton of this kind can form from sound waves propagating through plasma. These waves, unlike gas, are carried by ions that also interact with each other through an electric field. These are the best studied solitons in plasmas and are easily generated in *Double Plasma* (DP) apparatus by a voltage pulse of suitable frequency. Planar solitons, cylindrical solitons or spherical solitons can be created without problems. Ion-acoustic solitons are described by the KdV equation and behave according to it in experiments.

The DP apparatus is the most important piece of equipment that has contributed to soliton research. One of the many variants in use is shown in Fig. 65. The first device was built by Rudolf Limpaecher and K. R. MacKenzie in 1973. It is a double plasma chamber, the two parts being separated by a metal mesh. The latter is not electrically connected to any element of the apparatus, it spontaneously charges to a negative potential (few tens of volts) and prevents the passage of electrons from one volume to the other. The voltages ϕ_{HD} and ϕ_{HT} ensure the heating of the electrodes from which electrons escape into the chambers. The voltages ϕ_{AD} and ϕ_{AT} accelerate these electrons, which ionize the working gas. To prevent electrons

from escaping the chambers, there are rows of permanent dipole magnets on the surface (adjacent rows have opposite polarity), for huge chambers, there may be one to two thousand of them. A suitably chosen voltage ϕ_B prevents the passage of ions between the chambers.
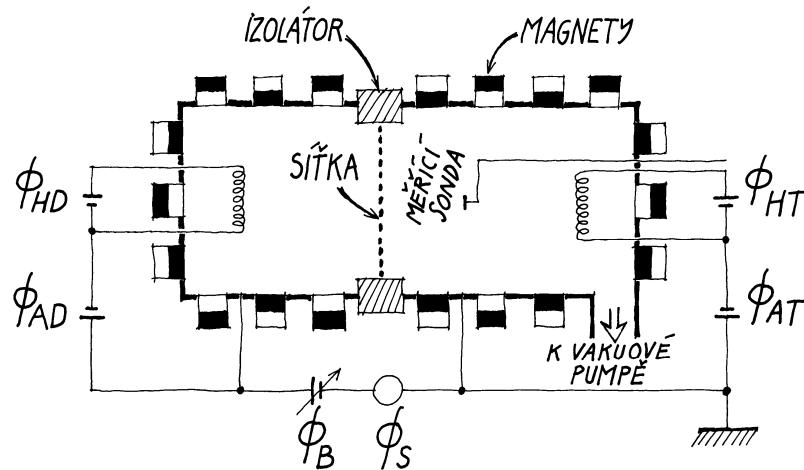
Double Plasma (DP) apparatus

In the first version of the device, it was possible to create approximately 1 m^3 of collisionless plasma (the mean free path of the particles was larger than the dimensions of the apparatus) with an electron concentration of 10^7 to 10^{10} particles per cm^3, an electron temperature of 1 to 3 eV, and an ion temperature of less than 0.1 T_e (the low ion temperature ensures that the generated ion-acoustic waves are not attenuated by ion Landau damping). The permanent magnets had an induction of 0.2 T and their rows were 10 cm apart. The soliton could be induced in a number of ways, one of which was to deliver a voltage pulse ϕ_S that resulted in a large perturbation of the ion density propagating from the preparation chamber to the target chamber (the chamber with the measuring probe). However, there are other ways: applying a voltage to the grid, applying a voltage to a metallic object immersed in the left preparation chamber (the cylindrical object formed a cylindrical soliton) or photoionization by light focused in a certain plane (a planar soliton was formed) or voltage applied to an array of wires. Devices of this type are the standard for the creation and research of solitons associated with ion-acoustic waves, and are now also produced with cylindrical or spherical chambers.

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6. TURBULENCE

Turbulent structures

In this chapter we will discuss another very important nonlinear phenomenon – turbulence. Turbulent motions are one of the most common motions in nature, and at the same time turbulence is a phenomenon that has not been satisfactorily solved to this day. It can be approached using dimensional analysis, magnetohydrodynamics, statistics, or numerical simulation of the phenomena. The following text is only a basic overview of some of the relationships, and is certainly not a full-fledged textbook. More detailed information can be found in specialized textbooks.

Turbulence most often occurs at high fluid flow velocities. Excess kinetic energy causes a transition from laminar flow to turbulent flow. Energy is stored by non-linear phenomena in the rotational motion of vortices of various sizes. However, turbulence is also encountered when obstacles are passed, for example behind ships or bridge piers. Turbulent eddies are typical of the atmosphere and of the impact of water in a cooking pot. In plasmas, we encounter turbulent motions very often. Turbulence is present in the solar wind and other astrophysical plasmas, and occurs in technological and fusion plasmas. In plasmas, turbulence is most often due to the development of various instabilities.

At first sight, turbulence is characterized by chaos, seemingly random fluctuations in the velocity and pressure fields caused by a transition to a state with a large number of vortices. However, the most characteristic phenomenon of turbulence is the so-called *energy cascade*. Vortices of various sizes are generated in the flow. The largest ones have dimensions comparable to the smallest dimension of the flow area. The large eddies stretch and break up into smaller eddies. Vortices of a certain size transfer energy by non-linear phenomena to smaller vortices, which in turn transfer energy to smaller ones, etc. Energy is transformed by the flowing medium from the largest structures to the smallest ones. The size of the vortices also has a lower limit – we call it the *Kolmogorov scale*. For small eddies the energy cascade ends and dissipative phenomena due to the viscosity of the fluid begin to dominate. The energy of the vortices is eventually converted into chaotic thermal energy. Turbulent processes also lead to diffusion of the substance; we speak of so-called turbulent diffusion.

The velocity field can be decomposed in a standard way into individual Fourier modes

$$\blacktriangleright \qquad\qquad \mathbf{u}(t,\mathbf{x}) = \int \mathcal{U}(\omega,\mathbf{k})\, e^{i[\mathbf{kx}-\omega t]}\, d^3\mathbf{k}\,. \qquad\qquad (334)$$

The four numbers ω, $\mathbf{k}$ of the Fourier modes are linked by a dispersion relation. The size of the vector $\mathbf{k}$ describes the dimensions of the structures corresponding to a given mode:

$$\blacktriangleright \qquad\qquad k = \frac{2\pi}{\lambda} \sim \frac{2\pi}{R} \qquad\qquad (335)$$

The largest eddies have a R_C dimension comparable to the described region, the smallest have a Kolmogorov scale of RK. In plasmas, the lowest limit of the size of the vortices is the ion Larmor radius R_{Li}. The so-called energy spectrum is important for turbulence. The kinetic energy of the vortices per unit mass is composed of the energies of the individual Fourier modes:

$$\blacktriangleright \qquad\qquad \langle\, \mathbf{u}\cdot\mathbf{u}\,/\,2\,\rangle = \int\limits_0^\infty \mathcal{E}(k)\, dk\,. \qquad\qquad (336)$$

The quantity $\mathcal{E}(k)$ is called the energy spectrum. In turbulence we distinguish three regions (according to the size of the vortices), see Figure 1.133, which differ in the shape of the energy spectrum:

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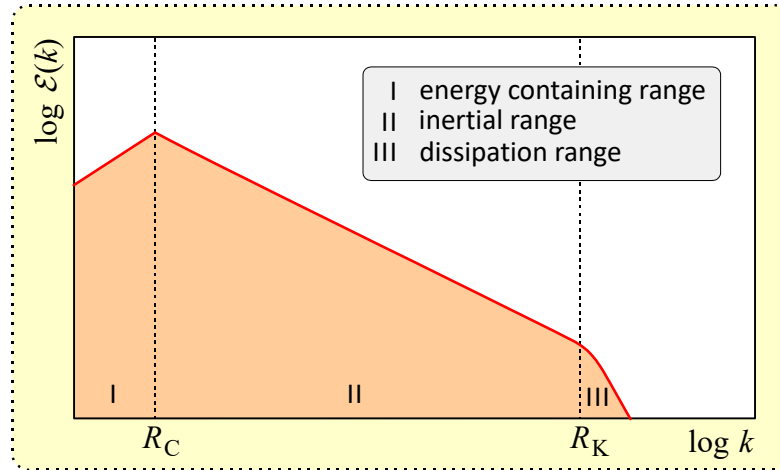
The three basic turbulent regimes

The largest vortices belong to area I, the so-called *energy containing range*. Viscous processes are negligible, and vortices transfer only little energy to smaller structures. Large vortices are neither homogeneously nor isotropically distributed. This region is insignificant for energy transfer. Typical for turbulence is region II, which we refer to as the *inertial range*. The turbulent energy cascade described above takes place here. Vortices of larger sizes transfer energy to vortices of smaller sizes. The viscosity of the fluid is important for these phenomena, but it does not dominate. For ordinary fluids, the distribution of vortices is homogeneous and isotropic, and the energy spectrum is proportional to $k^{-5/3}$ (the so-called Kolmogorov spectrum). The existence of this region and energy cascade distinguishes turbulence from other chaotic phenomena. In plasma, the vortices are affected by the magnetic field, which causes anisotropy in their distribution and a different dependence of the energy spectrum on the longitudinal component of the wave vector and a different one on the perpendicular component (with respect to the magnetic field). The last region is the *dissipation range* III, in which viscous phenomena dominate and small eddies disappear. Their energy is converted to heat.

Kolmogorov turbulence

The first successful model of turbulence was presented by the Soviet mathematician Andrei Nikolaevich Kolmogorov (1903–1987) in 1941. This turbulence model is usually referred to as K41. Kolmogorov assumed that all eddies of a turbulent energy cascade are spatially and temporally small compared to the dimensions of the region, i. e., their size satisfies $r \ll L$ and their time of change is $\delta t \ll L/u$. These eddies have no preferred orientation, i. e., they are homogeneously and isotropically distributed. All directional dependencies are lost in the course of the energy cascade, the resulting statistics of the vortices are of universal character and depend only on the energy dissipation rate $\varepsilon \equiv d(u^2/2)/dt$ given by the viscous processes and on the size of the vortices given by the wave number k. The resulting Kolmogorov relations can be easily derived from a simple dimensional analysis of the problem. We relate the energy per unit mass, so the kinetic energy of the vortices is $u^2/2$. We give the dimensions of the key quantities from relation (336) and the energy dissipation rate ε:

$$\left[u^2/2\right] = \mathrm{m^2 s^{-2}}; \qquad [k] = \mathrm{m^{-1}}; \qquad [\mathcal{E}] = \mathrm{m^3 s^{-2}}; \qquad [\varepsilon] = \mathrm{m^2 s^{-3}}. \tag{337}$$

A simple dimensional analysis will easily show that from these relations no other dependence is possible for the energy spectrum than

▶
$$\mathcal{E} = K_0 \varepsilon^{2/3} k^{-5/3}, \tag{338}$$

where K_0 is some universal dimensionless constant. This dependence governs many fluids in region II, where the turbulent energy cascade takes place. For plasmas, the fluctuations of the electric potential can be substantial, the spectrum of which is given by

$$\langle \, (\delta\phi)^2 \, \rangle = \int_0^\infty S(k)\,\mathrm{d}k\,. \tag{339}$$

The dimensional analysis here leads to dependence

$$S(k) \sim k^{-5}\,. \tag{340}$$

However, the above relation is valid only under the assumption that the fluctuation of the electric potential does not depend on the Larmor radius of the ions. The Larmor radius would introduce an additional quantity with a meter dimension into the dimensional analysis, and therefore the dimensional analysis would no longer give an unambiguous result. Thus, relation (340) should be taken only with a large cautiousness.

    Another quantity that can be estimated from dimensional analysis is the Kolmogorov scale RK. From the kinematic viscosity ν and the energy dissipation rate ε, a single quantity with length dimension can be composed. Therefore, it can be estimated that the Kolmogorov length is given by the following relation

$$R_{\mathrm{K}} \cong \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}\,. \tag{341}$$

Again, the relationship only holds for processes in which gyration ion movements do not play a major role. It can therefore be applied to turbulence of ordinary fluids without the presence of magnetic fields.

Turbulence in magnetohydrodynamics

Assume the simplest version of magnetohydrodynamics with viscous processes. The fluid is assumed to be incompressible. The initial equations will be

$$\blacktriangleright \qquad \rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \eta_{\mathrm{L}}\Delta\mathbf{u} - \nabla p_{\mathrm{L}} + \frac{1}{\mu}\operatorname{rot}\mathbf{B}\times\mathbf{B}\,, \tag{342}$$

$$\blacktriangleright \qquad \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\sigma\mu}\Delta\mathbf{B} + \operatorname{rot}\mathbf{u}\times\mathbf{B}\,. \tag{343}$$

The first equation is the equation of motion of the plasma, on the right are successively: the density of the viscous force, the density of the pressure force and the density of the Lorentz force. The second equation is the equation for the time evolution of the magnetic field, the first term describes the diffusion of the field, the second the freezing of the field. Both equations must be supplemented with equations for the divergence of the fields. For an incompressible fluid, we have simple equations

$$\operatorname{div}\mathbf{B} = 0\,, \tag{344}$$

$$\operatorname{div}\mathbf{u} = 0\,. \tag{345}$$

The relation (345) says that the plasma is incompressible, thus its density is constant. Therefore, we divide equation (342) by the density and comine it with the permeability into the magnetic field to obtain the relations

$$\blacktriangleright \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu_{\mathrm{L}}\Delta\mathbf{u} - \nabla P_{\mathrm{L}} + \operatorname{rot}\mathbf{b}\times\mathbf{b}\,, \tag{346}$$

▶
$$\frac{\partial \mathbf{b}}{\partial t} = \nu_M \Delta \mathbf{b} + \mathrm{rot}\, \mathbf{u} \times \mathbf{b} \,, \tag{347}$$

where we denoted

$$\mathbf{b} \equiv \frac{\mathbf{B}}{\sqrt{\mu\rho}}; \qquad P_L \equiv \frac{p_L}{\rho}; \qquad \nu_M \equiv \frac{1}{\sigma\mu} \tag{348}$$

The scaled magnetic field $\mathbf{b}$ has a velocity dimension – it is nothing else than the Alfvén velocity of the field. Thus the equations for the velocity and magnetic fields now have the same dimension. By comparing the different terms we can obtain dimensionless characteristics of the flow:

$$\#_{Re} \equiv \frac{|(\mathbf{u}\cdot\nabla)\mathbf{u}|}{|\nu_L \Delta \mathbf{u}|} \approx \frac{Lu}{\nu_L}; \tag{349}$$

$$\#_{ReM} \equiv \frac{|\mathrm{rot}\,\mathbf{u}\times\mathbf{b}|}{|\nu_M \Delta \mathbf{b}|} \approx \frac{Lu}{\nu_M}; \tag{350}$$

$$\#_{Pr} \equiv \frac{\#_{ReM}}{\#_{Re}} \approx \frac{\nu_M}{\nu_L}; \tag{351}$$

The Reynolds number $\#_{Re}$ expresses the ratio of the kinetic component to the viscous component. At high Reynolds numbers, there is an excess of kinetic energy in the flow and turbulent flow develops. The value of the Reynolds number is therefore decisive for the spontaneous development of turbulence. In plasmas this can of course be aided by the development of various instabilities. The magnetic Reynolds number $\#_{ReM}$ is the ratio of the freezing term and the diffusion term in the equation for the time evolution of the magnetic field and we have already encountered it. In most types of plasmas it is very high and the freezing of the field dominates over its diffusion. The magnitude of the magnetic Reynolds number does not determine the turbulence. The last one, the Prandtl number, puts the two types of viscous processes together – it is the ratio of magnetic diffusion and kinematic viscosity. In astrophysical plasmas, it can have a variety of values, from 10^{29} in intergalactic space to 1 in the solar wind to $\ll 1$ in the solar convective zone. For turbulent processes, it expresses the contribution of the influence of magnetic (ohmic) and viscous energy dissipation.

Elsässer fields

Equations (346) and (347) for the velocity and magnetic fields are further modified to symmetric form. In both equations, we decompose the double vector products on the right-hand side. For the first equation, the adjustment will lead to a splitting of the Lorentz force density into a magnetic pressure contribution and a magnetic field line curvature contribution. For the magnetic field equation, the equation will be converted to a full time derivative equation on the left-hand side. We have made both modifications before, so we will only give the result:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu_L \Delta\mathbf{u} - \nabla\left(P_L + P_M\right) + (\mathbf{b}\cdot\nabla)\mathbf{b} \,, \tag{352}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} = \nu_M \Delta\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} \,. \tag{353}$$

The magnetic pressure is described in the equation for the velocity field by the quantity

$$P_M \equiv \frac{p_M}{\rho} = \frac{\mathbf{B}^2}{2\mu\rho} = \frac{\mathbf{b}^2}{2} \tag{354}$$

Both equations (352), (353) have the same dimension, so we can combine them appropriately. In a system associated with flowing plasma, both fields will have the form

$$\mathbf{u} = \delta\mathbf{u} \, , \tag{355}$$

$$\mathbf{b} = \mathbf{b}_0 + \delta\mathbf{b} \, . \tag{356}$$

The velocity field has only a fluctuating part, the magnetic field is given by the ground and fluctuating fields. In 1950, the American physicist Walter Elsässer introduced new variables

$$\mathbf{z} = \delta\mathbf{u} - \delta\mathbf{b} \, , \tag{357}$$

$$\mathbf{w} = \delta\mathbf{u} + \delta\mathbf{b} \, . \tag{358}$$

Sometimes Elsässer fields are denoted only by the $\mathbf{z}_\pm$ symbols. To obtain the equations for the new fields, add and subtract equations (352), (353) for the velocity and magnetic field:

▶ $$\frac{\partial\mathbf{z}}{\partial t} + (\mathbf{b}_0\cdot\nabla)\mathbf{z} + (\mathbf{w}\cdot\nabla)\mathbf{z} = \Delta\left[\frac{\nu_{\mathrm{L}} - \nu_{\mathrm{M}}}{2}\mathbf{w} + \frac{\nu_{\mathrm{L}} + \nu_{\mathrm{M}}}{2}\mathbf{z}\right] - \nabla(P_{\mathrm{L}} + P_{\mathrm{M}}), \tag{359}$$

▶ $$\frac{\partial\mathbf{w}}{\partial t} - (\mathbf{b}_0\cdot\nabla)\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} = \Delta\left[\frac{\nu_{\mathrm{L}} + \nu_{\mathrm{M}}}{2}\mathbf{w} + \frac{\nu_{\mathrm{L}} - \nu_{\mathrm{M}}}{2}\mathbf{z}\right] - \nabla(P_{\mathrm{L}} + P_{\mathrm{M}}). \tag{360}$$

The equations for Elsässer fields are the initial equations for investigating magnetohydrodynamic turbulence, whether analytically, by dimensional analysis or numerical simulations. If we either neglect the magnetic field or assume that the fluctuations are homogeneous and isotropic and the typical time scale is given only by nonlinear phenomena, i. e.

$$\tau_{\mathrm{NL}} \simeq (kw)^{-1} \simeq (kz)^{-1} . \tag{361}$$

Either analytically or by dimensional analysis we obtain the Kolmogorov spectrum

▶ $$\mathcal{E}_{\mathrm{K}} \sim \varepsilon^{2/3}k^{-5/3} \tag{362}$$

However, we will be primarily interested in the influence of the background magnetic field, where the typical time scale is given by the Alfvén time

$$\tau_{\mathrm{A}} \simeq \frac{L}{v_{\mathrm{A}}} = \frac{L}{b_0} . \tag{363},$$

If we keep the assumption of isotropy of fluctuations in this case (it is justified for weak fields and small vortices), we get the so-called IK or IK64 model named after the Soviet physicist P. S Iroshnikov and the American theorist Robert Henry Kraichnan. Iroshnikov described turbulence in this way in 1964 and Kraichnan independently in 1965. The result is the spectral dependence

▶ $$\mathcal{E}_{\mathrm{IK}}(k) \sim (v_{\mathrm{A}}\varepsilon)^{1/2}k^{-3/2} . \tag{364}$$

The spectrum of the IK model therefore depends on the background magnetic field.

Strong anisotropic turbulence

The strong magnetic field introduces anisotropy into the plasma and the assumption of homogeneous and isotropic turbulence is untenable in this case. The energy spectrum, unlike the Kolmogorov turbulence (336), must be defined three-dimensionally:

$$\langle\mathbf{u}\cdot\mathbf{u}/2\rangle = \int_0^\infty \mathcal{E}(k_\parallel, k_\perp)\,\mathrm{d}^3\mathbf{k} . \tag{365}$$

The dimension of the energy spectrum is therefore different in three dimensions than in the one-dimensional Kolmogorov description. Anisotropic models can be divided into weakly

and strongly turbulent models. In weak turbulence, both Elsässer fields are much smaller than the background magnetic field, while in strong turbulence such an assumption does not hold. For weak turbulence (WT) an energy spectrum can be derived

▶
$$\mathcal{E}_{\text{WT}}(k) \sim k_{\parallel}^{1/2} k_{\perp}^{-2} . \tag{366}$$

However, the conditions of weak turbulence are rarely fulfilled in the plasma; if turbulence develops, it is usually strong. The conditions of strong magnetohydrodynamic turbulence have been studied by the American astrophysicist Peter Goldreich and the Indian physicist Seshadri Sridhar. Their 1995 results are referred to as the GS or GS95 model. The energy spectrum is very interesting. If the fluctuations are isotropic at least in the perpendicular plane to the magnetic field, the spectrum becomes one-dimensional and leads to a Kolmogorov shape:

▶
$$\mathcal{E}_{\text{GS}}(k) \sim \varepsilon^{2/3} k_{\perp}^{-5/3} . \tag{367}$$

For the parallel component, an equilibrium is established between the two components of the wave vector given by

▶
$$k_{\parallel} \sim k_{\perp}^{2/3} . \tag{368}$$

However, the coefficients of proportionality in the GS95 model are a function of Alfvén velocity and other parameters, making experimental verification somewhat difficult. Numerical simulations show that in the case of strongly developed turbulence with a background magnetic field, the results for weak fields are close to the GS model, while for strong fields they correspond to the IK model. The problem of turbulence is still an open discipline with many unanswered questions.
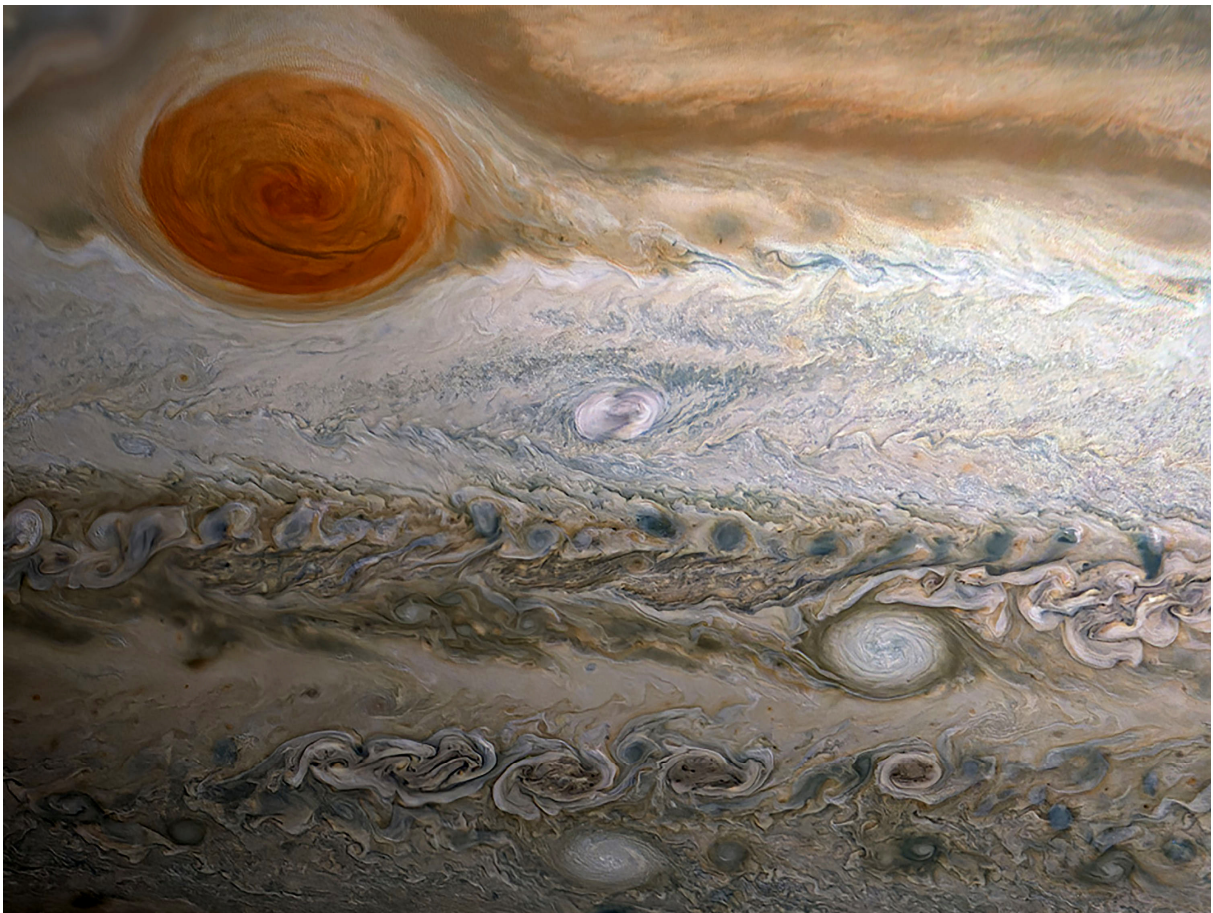
Turbulent structures in Jupiter's clouds. Photo: JUNO spacecraft, 2 June 2020, NASA/JPL.

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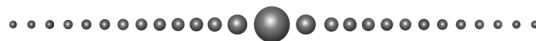
7. SELECTED PROBLEMS FROM PLASMA PHYSICS

Schwarzschild criterion

In 1906, the German physicist Karl Schwarzschild (1873–1916) derived a criterion for the development of convective flows inside stars. His criterion for the development of convection is valid not only in stars but also for any plasma system. Suppose that by some random process a small region of plasma moves to another location in the star. For the sake of certainty, consider a small region of plasma that moves from the interior of the star a little higher in the *z*-axis direction (see the following figure).
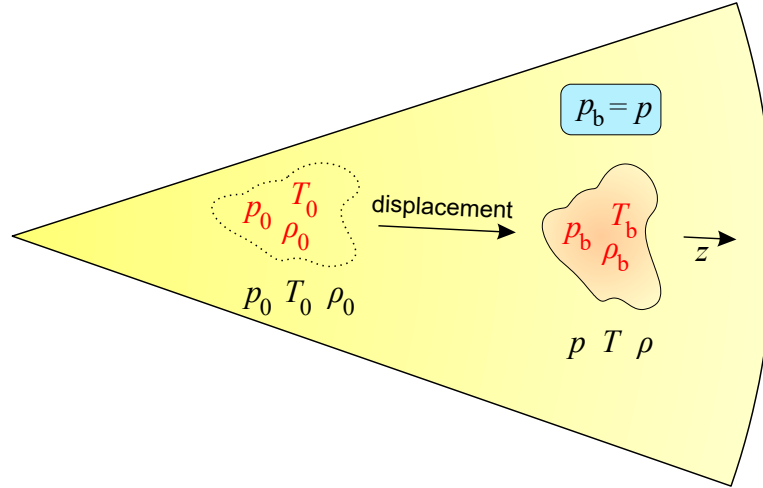
Before the transfer, it is just a thought region and the parameters of our "bubble" are the same as those of the surrounding plasma. But after the transfer the situation will be different, in the bubble there will be temperature T_b, density ρ_b and pressure p_b, in the surrounding plasma there will be values of T, ρ, p. Only the pressures have to match, otherwise the bubble boundary would run elsewhere. If the density of the bubble in the new region is greater than that of the surroundings, it will tend to go back (in the direction of gravity) and it will be a stable situation. If the density of the bubble is less than the surrounding area, it will continue to move in the same direction and convection (flow) will develop. We say that the situation is convective unstable. The condition for the development of a flow is therefore very simple:

▶
$$\rho_b < \rho . \tag{369}$$

First, let's discuss the left-hand side of the criterion. Assume that the bubble displacement is a classical disturbance, it takes a very short time and the bubble is not enough to exchange thermal energy with the surroundings, i. e. it is an adiabatic process:

$$p_b = K\rho^\gamma ; \qquad \gamma \equiv \frac{c_p}{c_V} . \tag{370}$$

We now carry out a Taylor expansion of this relation to first order:

$$p_b = K\rho^\gamma \approx p_0 + \frac{\partial p_b}{\partial \rho}(\rho_b - \rho_0) = p_0 + K\gamma\rho_0^{\gamma-1}(\rho_b - \rho_0) = p_0 + \frac{p_0}{\rho_0}\gamma(\rho_b - \rho_0). \tag{371}$$

We determined the constant K from relation (370). Now we can easily calculate the density of the bubble in the left part of the criterion (in the calculation we use that the pressure in the bubble is the same as in the surroundings):

$$\rho_b = \rho_0 + \frac{\rho_0}{\gamma p_0}(p - p_0). \tag{372}$$

The right-hand side of the criterion is expressed from the plasma equation of state

$$p = A\rho T \tag{373}$$

We will perform expansion to first order and express the constant A from the last relation:

$$p = p_0 + \frac{\partial p}{\partial \rho}(\rho - \rho_0) + \frac{\partial p}{\partial T}(T - T_0) = p_0 + AT_0(\rho - \rho_0) + A\rho_0(T - T_0) =$$

$$= p_0 + \frac{p_0}{\rho_0}(\rho - \rho_0) + \frac{p_0}{T_0}(T - T_0) \tag{374}$$

Now we determine the density in the first part of the criterion:

$$\rho = \rho_0 + \frac{\rho_0}{p_0}\left(p - p_0\right) - \frac{\rho_0}{T_0}(T - T_0) \tag{375}$$

After substituting (375) and (372) into the criterion (369) we have the condition for the development of convection

$$\frac{\rho_0}{\gamma p_0}(p - p_0) < \frac{\rho_0}{p_0}\left(p - p_0\right) - \frac{\rho_0}{T_0}(T - T_0). \tag{376}$$

The following modifications are already straightforward and lead to the relationship

$$\mathrm{d}T < \left(\frac{\gamma - 1}{\gamma}\right)\frac{T_0}{p_0}\,\mathrm{d}p. \tag{377}$$

But for the situation shown in the figure, both $\mathrm{d}T$ and $\mathrm{d}p$ are negative (both pressure and temperature decrease with increasing height), so the resulting relationship is (we can omit the subscripts 0, this is the general plasma n in which we did the thought experiment):

$$\blacktriangleright \qquad |\mathrm{d}T| > \left(\frac{\gamma - 1}{\gamma}\right)\frac{T}{p}|\mathrm{d}p|. \tag{378}$$

For convection to develop, a sufficiently large temperature gradient is required in the plasma, which is met, for example, in our Sun up to 200,000 km below the surface and above. Energy from the Sun's interior is first transferred by radiation and only for the last 200,000 km by flow. In stars with a CNO cycle in their interior, the energy production in the core is much higher than in our Sun, and the temperature gradient is, in turn, greatest around the core. In these stars, convection develops only around the core and energy is transferred only by radiation in the outer layers. The Schwarzschild criterion is given in various alternative forms, to name a few:

$$\blacktriangleright \qquad \frac{|\mathrm{d}T|}{T} > \left(\frac{\gamma - 1}{\gamma}\right)\frac{|\mathrm{d}p|}{p}, \tag{379}$$

$$\blacktriangleright \qquad \frac{\mathrm{d}\ln T}{\mathrm{d}\ln p} < \frac{\gamma - 1}{\gamma}. \tag{380}$$

The Schwarzschild criterion applies in general, and where there is a large temperature gradient, an unstable situation and flow development will occur in the plasma.

Relativistic Boltzmann equation

The Boltzmann equation is used in plasma physics to determine the space and time evolution of the particle probability density and can be written, including normalization, as

$$f = f(t, \mathbf{x}, \mathbf{v}); \tag{381}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = S; \tag{382}$$

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}}) f + \left(\frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} \right) f = S \,, \tag{383}$$

$$\int f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}^3 \mathbf{v} = n(t, \mathbf{x}) \,. \tag{384}$$

The probability density is normalized with respect to the number of particles. The left-hand side of equation (383) is clearly non-relativistic, as is the collisional term on the right-hand side. The collision term can be expressed in many different forms, one of the most commonly used is the form leading to the Fokker-Planck equation. There is no general transcription of the collision term into relativity. In some cases it is not even necessary to know it, since at high velocities the effective cross section of the precipitation usually decreases rapidly and can be neglected. However, this is not universally true, and a relativistic transcription of the collisional term would be very useful in some cases. The situation is simpler with the left-hand side of the Boltzmann equation, which can easily be rewritten into a Lorentzian covariant form. Instead of velocity as the independent variable, we must use momentum, or four-momentum, i. e.

$$f = f(x^\alpha, P^\alpha) = f(t, \mathbf{x}, E, \mathbf{p}) \,; \tag{385}$$

The four-momentum has interdependent components (the magnitude of the four-momentum is $-m_0^2 c^2$), so that

$$E^2 = \mathbf{p}^2 c^2 + m_0^2 c^4 \,, \tag{386}$$

and therefore $E = E(\mathbf{p})$. The new normalization will refer to momentum instead of velocity, so we will require

$$\int f(x^\alpha, P^\alpha) \, \mathrm{d}^3 p = n(t, \mathbf{x}) \,. \tag{387}$$

Such a notation looks non-relativistic at first sight, but we have to remember that $E = E(\mathbf{p})$ and relativistically the normalization can be written over the Dirac distribution:

▶ $$\int f(t, \mathbf{x}, E, \mathbf{p}) \, \delta(E - E(\mathbf{p})) \, \mathrm{d}^4 P = n(t, \mathbf{x}) \,. \tag{388}$$

We have to generalize the relation (382) relativistically, so instead of the coordinate time we use the proper time, which is the invariant of the Lorentz transformation, i. e.

$$\frac{\mathrm{d} f}{\mathrm{d} \tau} = S \,; \tag{389}$$

After elaborating the derivation on the left, we have

$$\frac{\partial f}{\partial x^\alpha} \frac{\mathrm{d} x^\alpha}{\mathrm{d} \tau} + \frac{\partial f}{\partial P^\alpha} \frac{\mathrm{d} P^\alpha}{\mathrm{d} \tau} = S \,. \tag{390}$$

The first term contains both time and space derivatives and is a generalization of the first two terms on the left side of the Boltzmann equation. Realizing the importance of both time derivatives, we can write

$$U^\alpha \frac{\partial f}{\partial x^\alpha} + F^\alpha \frac{\partial f}{\partial P^\alpha} = S \,. \tag{391}$$

We express the Lorentz four-force from the relation (101)

$$U^\alpha \frac{\partial f}{\partial x^\alpha} + Q F^{\alpha\beta} U_\beta \frac{\partial f}{\partial P^\alpha} = S \,. \tag{392}$$

As a final step, we need to convert the four-velocities into four-momenta:

▶ $$P^\alpha \frac{\partial f}{\partial x^\alpha} + Q F^{\alpha\beta} P_\beta \frac{\partial f}{\partial P^\alpha} = m_0 S \,. \tag{393}$$

This is the desired relativistic covariant notation of the Boltzmann equation. In the first term, we can move the four-momentum to the probability density and nothing happens. But in the second term, we can do this as well, because

$$\frac{\partial}{\partial P^\alpha}\left(F^{\alpha\beta}P_\beta f\right) = F^{\alpha\beta}\delta_{\alpha\beta}f + F^{\alpha\beta}P_\beta\frac{\partial f}{\partial P^\alpha} = F^{\alpha\beta}P_\beta\frac{\partial f}{\partial P^\alpha}. \tag{394}$$

The first term is zero after the derivation of the product because it is a contraction of the symmetric and antisymmetric matrix. However, this will not be true for a general velocity-dependent force field. So we can write

$$\blacktriangleright \qquad \frac{\partial}{\partial x^\alpha}\left(P^\alpha f\right) + \frac{\partial}{\partial P^\alpha}\left(QF^{\alpha\beta}P_\beta f\right) = m_0 S. \tag{395}$$

This form is an elegant notation of the Boltzmann equation for the case of the electromagnetic field and the Lorentz four-force.

<p align="center">* * *</p>

An interesting problem is the rewriting of the Boltzmann equation in general relativity, where the motion of particles is given by the curvature of spacetime. In this case, we introduce a covariant derivative of the four-vector

$$\frac{DA^\alpha}{Dx^\nu} \equiv A^\alpha{}_{;\nu} = A^\alpha{}_{,\nu} + \Gamma^\alpha_{\mu\nu}A^\mu, \tag{396}$$

where the first term is an ordinary partial derivative and the second term is related to the change in the four-vector A^α caused by the curvature of spacetime. The coefficients Γ are called Christofell symbols and are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right). \tag{397}$$

The geodesics equation (straightest possible path in curved spacetime) is

$$\mathrm{d}^2 x^\alpha + \Gamma^\alpha_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = 0. \tag{398}$$

We express the four-force in equation (391) from the geodesics equation:

$$F^\alpha = \frac{\mathrm{d}P^\alpha}{\mathrm{d}\tau} = m_0\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = m_0\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = -m_0\,\Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -m_0\Gamma^\alpha_{\mu\nu}U^\mu U^\nu. \tag{399}$$

Boltzmann's equation (391), when rewritten in four-momentum terms, takes the form

$$\blacktriangleright \qquad P^\alpha\frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\mu\nu}P^\mu P^\nu\frac{\partial f}{\partial P^\alpha} = m_0 S. \tag{400}$$

Ray-Tracing

A very interesting task is to track a beam as it moves through a plasma medium (in the geometrical optics approximation). There are sophisticated methods to determine the direction of the beam at a given location and its movement to the next location from knowledge of the permeability tensor and permittivity tensor. However, such a problem assumes knowledge of both tensors. The ray tracing can also be done easily from the mere knowledge of the dispersion relation for the propagation of an electromagnetic wave. We start with Hamilton's equations for the quantum of electromagnetic radiation

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{401}$$

We will now use the relations for particle-wave dualism

$$H = \hbar\omega; \qquad \mathbf{p} = \hbar\mathbf{k} \tag{402}$$

and insert both relations into Hamilton's equations:

▶
$$\frac{d\mathbf{x}}{dt} = \frac{\partial\omega}{\partial\mathbf{k}}, \qquad \frac{d\mathbf{k}}{dt} = -\frac{\partial\omega}{\partial\mathbf{x}}. \tag{403}$$

The equations found can be used for ray tracing. But it is necessary to remember a few basic facts: 1) time is here only as a parameter of the ray trajectory; 2) when performing a numerical simulation we always get at a given point $\mathbf{x}$ a vector $\mathbf{k}$, i. e. a tangent to the ray to be traced. After that we move to the next point and repeat the procedure; 3) the found equations for ray-tracing require explicit knowledge of the dispersion relation $\omega = \omega(\mathbf{x}, \mathbf{k})$; i. e. from the dispersion relation we have to be able to express the angular frequency, which can sometimes be a problem; 4) the dependence on the position of $\mathbf{x}$ is obtained in the dispersion relation by knowing the dependence of the concentration or external magnetic (electric) field on the position in the plasma.

　　Most often we know the dispersion relation in implicit form, i.e. not solved with respect to the variables ω or $\mathbf{k}$:

$$\phi(\mathbf{x}, \omega, \mathbf{k}) = 0. \tag{404}$$

Let's find the differential of this session (frequency ω is fixed, given by the frequency of the transmitted beam)

$$\frac{\partial\phi}{\partial\mathbf{x}} \cdot d\mathbf{x} + \frac{\partial\phi}{\partial\mathbf{k}} \cdot d\mathbf{k} = 0. \tag{405}$$

Assume that the beam path is given parametrically, i. e.

$$\mathbf{x} = \mathbf{x}(\tau); \tag{406}$$
$$\mathbf{k} = \mathbf{k}(\tau).$$

From relation (405) we immediately have

$$\frac{\partial\phi}{\partial\mathbf{x}} \cdot \frac{d\mathbf{x}}{d\tau} + \frac{\partial\phi}{\partial\mathbf{k}} \cdot \frac{d\mathbf{k}}{d\tau} = 0. \tag{407}$$

The parameter τ can be arbitrary. We take advantage of this and choose it so that

▶
$$\frac{d\mathbf{x}}{d\tau} = \frac{\partial\phi}{\partial\mathbf{k}};$$
$$\frac{d\mathbf{k}}{d\tau} = -\frac{\partial\phi}{\partial\mathbf{x}}. \tag{408}$$

This choice both satisfies equation (407) and yields new equations for ray-tracing in which the role of the Hamiltonian is played by the function ϕ, i. e., an arbitrary implicit notation of the dispersion relation. An arbitrary differential scheme can be applied to the system of Hamiltonian equations (408) and we can conveniently track the beam passing through the plasma in some numerical way.

Mechanisms of current genesis in z-pinch

The Z-pinch is a simple configuration in which a cylinder of plasma is flowed by an electric current that generates an azimuthal magnetic field, the corresponding force of which keeps the plasma in (mostly unstable) equilibrium. The idea that the current is driven by electrons flying along the axis is completely wrong. The electrons are doing gyration motion and the drifts (curvature drift and grad B drift) and the magnetizing current, which is due to the Lar-

mor rotation of the particles, which is not compensated by the neighboring particles to exactly zero, are responsible for the electric current passing through the plasma in equilibrium. Let us now derive the relations for the individual current densities inside the filament, whose geometry is described by the coordinates shown in the following figure:
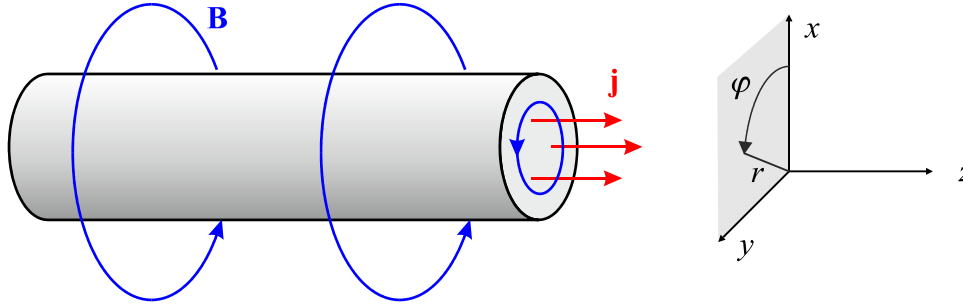
Gradient | B | drift. This drift is caused by a change in the density of the magnetic field lines and is related to the perpendicular component of the velocity of the charged particle (with respect to the magnetic field line). The electric current is given by the general formula

$$\mathbf{j}_{\nabla B} = \left\langle \sum_{\alpha} Q_{\alpha} n_{\alpha} \mathbf{v}_{\alpha} \right\rangle . \tag{409}$$

where summation is over electrons and ions, averaging over all particles. For the velocity we substitute the drift velocity

$$\mathbf{v}_{\nabla B} = \frac{-\mu \nabla B \times \mathbf{B}}{Q B^2} = \frac{m v_{\perp}^2}{2Q} \frac{\mathbf{B} \times \nabla B}{B^3} \tag{410}$$

and take advantage of the cylindrical symmetry of the current filament:

$$\mathbf{j}_{\nabla B} = -\frac{1}{B^2} \left\langle n_{\mathrm{e}} \frac{m_{\mathrm{e}} v_{\mathrm{e}\perp}^2}{2} + n_{\mathrm{i}} \frac{m_{\mathrm{i}} v_{\mathrm{i}\perp}^2}{2} \right\rangle \frac{\partial B}{\partial r} \mathbf{e}_z . \tag{411}$$

Le us recall that the field inside the fiber grows with increasing r and hence the derivative $\partial B/\partial r > 0$. From the geometry of the problem, it is clear that the grad B drift points in the negative direction of the z-axis. Let us now average the perpendicular component of the kinetic energy. The perpendicular component has only two degrees of freedom, and therefore

$$\left\langle \frac{m v_{\perp}^2}{2} \right\rangle = 2 \cdot \frac{1}{2} k_{\mathrm{B}} T = k_{\mathrm{B}} T , \tag{412}$$

and so

▶ $$\mathbf{j}_{\nabla B} = -\frac{1}{B^2} \left(n_{\mathrm{e}} k_{\mathrm{B}} T_{\mathrm{e}} + n_{\mathrm{i}} k_{\mathrm{B}} T_{\mathrm{i}} \right) \frac{\partial B}{\partial r} \mathbf{e}_z \;=\; -\frac{p}{B^2} \frac{\partial B}{\partial r} \mathbf{e}_z \; . \tag{413}$$

Curvature drift. This drift is due to the curvature of the magnetic field lines and is related to the longitudinal component of the velocity of the charged particle (relative to the magnetic field line). The electric current is again given by the general formula

$$\mathbf{j}_R = \left\langle \sum_{\alpha} Q_{\alpha} n_{\alpha} \mathbf{v}_{\alpha} \right\rangle . \tag{414}$$

For curvature drift, a simple relationship holds

$$\mathbf{v}_R = \frac{m v_\parallel^2}{Q B^2} \frac{\mathbf{R}_k \times \mathbf{B}}{R_k^2} . \tag{415}$$

Similar to the grad B drift, determine the current density from the curvature drift

$$\mathbf{j}_R = \frac{1}{rB} \left\langle n_e m_e v_{e\parallel}^2 + n_i m_i v_{i\parallel}^2 \right\rangle \mathbf{e}_z . \tag{416}$$

Calculate the mean value of the kinetic energy component (the particle has one degree of freedom along the magnetic field)

$$\left\langle \frac{m v_\parallel^2}{2} \right\rangle = 1 \cdot \frac{1}{2} k_B T = \frac{1}{2} k_B T \tag{417}$$

and for the current density due to curvature drift we have the resulting relation

▶
$$\mathbf{j}_R = \frac{1}{rB} \left(n_e k_B T_e + n_i k_B T_i \right) \mathbf{e}_z = \frac{p}{rB} \mathbf{e}_z . \tag{418}$$

Magnetization current. In the case of homogeneous plasma and a constant magnetic field, the current contribution from a system of identically Larmor-rotating particles is zero. If the field is inhomogeneous, the Larmor orbits are different at different locations and the average contribution to the flowing current may be non-zero. Similarly, in inhomogeneous plasma, the number of charge carriers increases in some direction and averaging the contribution to the total current yields a non-zero result. The magnetic moment of one particle is

$$\mathbf{p}_M = \frac{m v_\perp^2}{2B} \mathbf{e}_\varphi . \tag{419}$$

The gyrating charged particle generates its own magnetic field, which has the opposite direction to the original field. We therefore speak of plasma diamagnetism. In the coordinate system shown in the figure, the original magnetic field has direction $-\mathbf{e}_\varphi$, the magnetic moment of the particle has direction $+\mathbf{e}_\varphi$. We now determine the total magnetization and again center over the squared velocities:

$$\mathbf{M} = \left\langle \sum_\alpha n_\alpha \mathbf{p}_\alpha^{(M)} \right\rangle = \frac{\left\langle n_e m_e v_{e\perp}^2 + n_i m_i v_{i\perp}^2 \right\rangle}{2B} \mathbf{e}_\varphi = \frac{\left(n_e k_B T_e + n_i k_B T_i \right)}{B} \mathbf{e}_\varphi = \frac{p}{B} \mathbf{e}_\varphi . \tag{420}$$

The magnetization current is easily determined in the given geometry:

$$\mathbf{j}_M = \operatorname{rot} \mathbf{M} = -\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{p}{B} \right) \mathbf{e}_z . \tag{421}$$

Finally, let us show that the sum of the three current densities derived above gives the total current flowing through the plasma:

$$j_{\nabla B} + j_R + j_M = -\frac{p}{B^2} \frac{\partial B}{\partial r} + \frac{p}{rB} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{p}{B} \right) = -\frac{1}{B} \frac{\partial p}{\partial r} . \tag{422}$$

So obviously

$$\left(j_{\nabla B} + j_R + j_M \right) B = -\frac{\partial p}{\partial r} , \tag{423}$$

which is the condition of equilibrium $\mathbf{j} \times \mathbf{B} = -\nabla p$, in which the total current comes out. The microscopic processes are thus naturally linked to the macroscopic currents in the continuum

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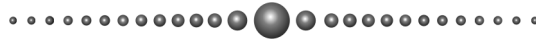
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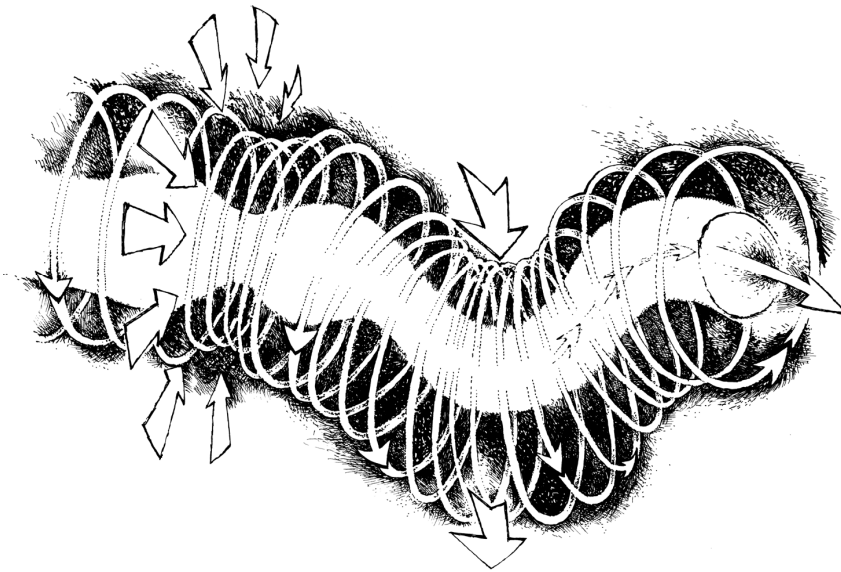
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